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Nonlocal Partial Differential Equations: Mathematical Theory, Numerical Approximation, and Applications in Multiscale Systems

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Abstract

Nonlocal partial differential equations (NPDEs) have emerged as an important mathematical framework for describing physical, biological, engineering, and environmental processes in which interactions extend beyond infinitesimally small neighbourhoods. Unlike classical partial differential equations, which generally describe local interactions through derivatives evaluated at individual points, nonlocal models incorporate spatially extended interactions through integral operators, convolution kernels, fractional derivatives, and other nonlocal mathematical formulations. These approaches provide an effective means of representing anomalous diffusion, long-range dependence, heterogeneous transport, memory effects, and multiscale phenomena that cannot always be adequately captured using conventional differential equations. This review presents a comprehensive examination of the mathematical foundations, theoretical developments, numerical approximation techniques, and interdisciplinary applications of nonlocal partial differential equations. Particular attention is given to fractional Laplacian operators, integro-differential equations, peridynamic formulations, nonlocal diffusion models, and equations incorporating spatial and temporal memory. The fundamental properties of existence, uniqueness, regularity, stability, and well-posedness are discussed in relation to different classes of nonlocal operators. Furthermore, numerical methodologies, including finite difference methods, finite element approximations, spectral techniques, discontinuous Galerkin methods, and fast convolution algorithms, are critically examined with respect to their accuracy, computational efficiency, and convergence behaviour. The application of nonlocal mathematical models in anomalous transport, fluid mechanics, material science, biological systems, image processing, environmental modelling, and complex networks is also explored. Particular emphasis is placed on multiscale systems, where nonlocal interactions facilitate the representation of processes operating across different spatial and temporal scales. Finally, current computational challenges and emerging research directions involving adaptive discretisation, machine learning-assisted numerical solvers, uncertainty quantification, and physics-informed neural networks are identified.

Keywords: *Nonlocal partial differential equations; Fractional calculus; Fractional Laplacian; Integro-differential equations.*

1. Introduction

Partial differential equations (PDEs) constitute one of the fundamental mathematical frameworks for describing the behaviour of physical, biological, chemical, and engineering systems. They provide mathematical representations of processes involving spatial and temporal variations, including heat transfer, fluid motion, wave propagation, diffusion, elasticity, electromagnetic interactions, and population dynamics. Classical PDEs generally rely on the assumption that interactions occurring at a particular location are predominantly determined by the immediate neighbourhood of that location. This local formulation has enabled the development of powerful mathematical theories and computational methods, including finite difference, finite element, and spectral approximation techniques. However, numerous natural and technological processes exhibit interactions extending over considerable spatial distances or depending on the historical behaviour of the system. Such phenomena may involve long-range correlations, heterogeneous transport mechanisms, memory-dependent responses, and anomalous diffusion, which cannot always be represented accurately through conventional local differential operators [1-2]. Nonlocal partial differential equations have developed as an important alternative for addressing these limitations. In contrast to classical differential equations, nonlocal models incorporate information from a finite or potentially unbounded region surrounding a given spatial point. Their mathematical formulations frequently involve integral operators, fractional derivatives, convolution kernels, and interaction functions that account for spatially

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extended dependencies. Consequently, the state of a system at a particular location may depend not only on the conditions at that location but also on the distribution of relevant quantities throughout a surrounding interaction domain. This characteristic makes nonlocal mathematical formulations particularly suitable for modelling systems in which long-range interactions, spatial heterogeneity, and multiscale dependencies play important roles. The mathematical foundations of nonlocal equations are closely associated with developments in fractional calculus, integral equations, functional analysis, and the theory of integro-differential operators. Fractional derivatives provide a particularly important class of nonlocal operators because they allow differentiation and integration of non-integer order. The fractional Laplacian, for example, extends the conventional Laplace operator and is widely employed in mathematical models of anomalous diffusion, stochastic processes, and long-range interactions. Similarly, peridynamic formulations replace classical spatial derivatives with integral expressions, allowing mechanical systems containing discontinuities and material fractures to be analysed without directly evaluating spatial derivatives at singular points [4-6]. One of the major motivations for adopting nonlocal models is their ability to describe anomalous transport processes. In classical diffusion, the mean-square displacement of particles generally increases linearly with time. However, experimental observations in porous media, biological tissues, heterogeneous materials, and complex fluids frequently reveal deviations from this behaviour. Subdiffusive and superdiffusive transport may arise from trapping mechanisms, heterogeneous environments, long-range particle movements, and temporal memory effects. Fractional and nonlocal diffusion equations provide mathematical tools for incorporating these mechanisms into transport models. Another important area of development involves multiscale systems. Many physical and biological processes are governed by interactions occurring at different spatial and temporal scales. For example, heat and mass transport in heterogeneous materials may depend on microscopic structural characteristics and macroscopic boundary conditions simultaneously. Similarly, biological populations may exhibit local movement, long-distance dispersal, and environmental interactions across extended geographical regions. Classical local models may require extensive modifications or computationally expensive multiscale formulations to represent these mechanisms. Nonlocal operators provide an alternative by incorporating spatially extended interactions directly into the governing equations. The development of efficient numerical methods remains an important research priority. Conventional discretisation approaches may become computationally expensive when applied to dense nonlocal interaction matrices [7-8]. Therefore, researchers have investigated fast convolution algorithms, sparse approximations, hierarchical matrices, fast Fourier transforms, adaptive mesh refinement, and domain decomposition techniques. More recently, machine learning and physics-informed neural networks have provided additional opportunities for approximating solutions of complex nonlocal systems.

2. Mathematical Foundations of Nonlocal Partial Differential Equations

2.1 Concept and General Mathematical Formulation

A nonlocal partial differential equation describes the evolution or equilibrium of a mathematical system in which the behaviour at a spatial location depends on information distributed over an extended region. The general structure of a nonlocal evolution equation can be expressed as

$$\frac{\partial u(x,t)}{\partial t} = \mathcal{L}_{NL} u(x,t) + f(x,t),$$

where $u(x,t)$ represents the unknown function, x denotes the spatial coordinate, t represents time, $f(x,t)$ is an external source term, and $\mathcal{L}_{NL}$ denotes a nonlocal operator. A widely used class of nonlocal operators is represented through an integral formulation:

$$\mathcal{L}_{NL} u(x) = \int_{\Omega} K(x,y)[u(y) - u(x)]\, dy,$$

where $K(x,y)$ is a nonlocal interaction kernel describing the strength of interaction between spatial points x and y, while Ω represents the spatial domain.

The interaction kernel determines the spatial range and intensity of nonlocal dependencies. Depending on its mathematical properties, the kernel may describe short-range interactions, long-range interactions, anisotropic behaviour, or spatially heterogeneous coupling. When the kernel is symmetric and nonnegative, the corresponding operator frequently possesses useful properties associated with energy dissipation and stability. The characteristic interaction distance is another important parameter in nonlocal modelling. A finite-horizon interaction kernel can be represented as

$$K_{\delta}(x,y) = 0 \quad \text{when} \quad |x-y| > \delta,$$

where δ represents the interaction horizon. This parameter determines the maximum distance over which spatial interactions are considered. Nonlocal models therefore provide a flexible mathematical framework in which the interaction range can be adjusted according to the physical or biological mechanisms being investigated.

2.2 Fractional Differential Operators

Fractional calculus provides one of the most extensively developed mathematical foundations for nonlocal PDEs. Unlike classical differentiation, which involves integer-order derivatives, fractional calculus generalises differentiation and integration to arbitrary real or complex orders. For a sufficiently regular function $u(x)$, the fractional Laplacian of order s, where $0 < s < 1$, can be defined as

$$(-\Delta)^s u(x) = C_{n,s}\, \mathrm{PV} \int_{\mathbb{R}^n} \frac{u(x) - u(y)}{|x-y|^{n+2s}}\, dy,$$

where n represents the spatial dimension, $C_{n,s}$ is a normalisation constant, and PV denotes the Cauchy principal value. The fractional Laplacian generalises the conventional Laplace operator and incorporates long-range spatial interactions through its singular integral representation. Its Fourier transform satisfies

$$\mathcal{F}[(-\Delta)^s u](\xi) = |\xi|^{2s}\hat{u}(\xi).$$

When s approaches unity, the fractional Laplacian converges, under suitable conditions, to the classical negative Laplacian. Fractional differential operators are particularly useful in modelling anomalous diffusion, stochastic processes, complex transport mechanisms, and systems involving spatial correlations across different scales.

2.3 Integro-Differential Equations

Integro-differential equations combine differential and integral operators within a single mathematical formulation. These equations are particularly suitable for describing systems in which local evolution is influenced by spatially extended interactions.

A representative nonlocal integro-differential equation can be expressed as

$$\frac{\partial u}{\partial t} = D\Delta u + \int_{\Omega} K\,(x,y)[u(y,t) - u(x,t)]\,dy + f(x,t),$$

where D represents the local diffusion coefficient. The first term describes conventional local diffusion, whereas the integral term accounts for nonlocal transport and spatial coupling. This combined formulation is useful in heterogeneous environments where conventional diffusion and long-range transport occur simultaneously. It also allows the mathematical representation of systems transitioning between local and nonlocal behaviour.

2.4 Nonlocal Models with Temporal Memory

Nonlocality may also occur in the temporal dimension. In memory-dependent systems, the present state depends on previous states rather than exclusively on the instantaneous configuration. A general memory-dependent evolution equation may be expressed as

$$\frac{\partial u(x,t)}{\partial t} = \int_0^t M\,(t-\tau)\mathcal{L}u(x,\tau)\,d\tau + f(x,t),$$

where $M(t-\tau)$ is a memory kernel describing the influence of past system states. When the memory kernel exhibits a power-law structure, fractional derivatives can provide an appropriate mathematical representation of the corresponding temporal dependence. Such models are particularly relevant to viscoelastic materials, biological transport, groundwater movement, and systems exhibiting trapping or relaxation phenomena.

3. Classification of Nonlocal Partial Differential Equations

Nonlocal PDEs can be classified according to the mathematical structure of their operators, the nature of their interaction kernels, and the physical mechanisms they represent.

3.1 Classification Based on Mathematical Operators

Table 1. Major classes of nonlocal partial differential equations and their mathematical characteristics

Type of equation	Mathematical formulation	Principal characteristics
Fractional diffusion equations	$\partial_t u + (-\Delta)^s u = f$	Long-range spatial transport
Nonlocal integro-differential equations	$\partial_t u = \int K(x,y)[u(y) - u(x)]dy + f$	Integral-based spatial interactions
Peridynamic equations	$\rho u_{tt} = \int H(u(y) - u(x), y-x)dy + f$	Nonlocal mechanical interactions
Fractional reaction-diffusion equations	$\partial_t u = -D(-\Delta)^s u + R(u)$	Nonlocal diffusion with nonlinear reactions
Time-fractional diffusion equations	${}^{C}D_t^{\alpha} u = D\Delta u + f$	Temporal memory and anomalous transport
Nonlocal conservation equations	$\partial_t u + \nabla \cdot F_{NL}(u) = 0$	Spatially extended transport and flux

The classification demonstrates that nonlocality can enter mathematical models through different mechanisms. Some formulations incorporate spatially extended interactions, whereas others introduce temporal memory or combine both characteristics.

3.2 Classification Based on Interaction Kernels

The interaction kernel plays a fundamental role in determining the mathematical and physical properties of nonlocal systems.

Common kernel structures include:

1. **Compact-support kernels:** Interactions are restricted to a finite neighbourhood.
2. **Power-law kernels:** Interaction intensity decreases according to a power-law relationship with distance.
3. **Gaussian kernels:** Interaction strength decreases rapidly with increasing spatial separation.
4. **Anisotropic kernels:** Interaction intensity depends on spatial direction.
5. **Heterogeneous kernels:** Interaction properties vary across the spatial domain.

The choice of kernel influences the regularity of solutions, convergence behaviour, computational requirements, and physical interpretation of the resulting mathematical model.

3.3 Local-to-Nonlocal Transition

An important theoretical question concerns the relationship between nonlocal operators and their classical local counterparts. For appropriately scaled interaction kernels, a nonlocal diffusion operator may converge to the classical Laplacian as the interaction horizon approaches zero. A simplified asymptotic representation is

$$\lim_{\delta \to 0} \mathcal{L}_{\delta} u(x) = c\Delta u(x),$$

where c is a scaling constant determined by the kernel properties. This relationship is important because it establishes a mathematical connection between classical PDEs and nonlocal formulations. It also provides a theoretical foundation for selecting interaction horizons in numerical models.

4. Theoretical Analysis of Nonlocal Partial Differential Equations

4.1 Existence and Uniqueness of Solutions

The existence and uniqueness of solutions represent fundamental requirements for establishing the mathematical reliability of nonlocal models. The analysis generally depends on the properties of the interaction kernel, the spatial domain, the initial and exterior conditions, and the function spaces in which solutions are defined. For linear nonlocal evolution equations, existence results may be established using semigroup theory, variational methods, and functional analytic techniques. Consider the abstract evolution equation

$$\frac{du}{dt} + Au = f(t),$$

where A represents a nonlocal operator acting on an appropriate Hilbert or Banach space. If the operator satisfies suitable conditions, such as closedness, coercivity, or appropriate dissipativity properties, mathematical techniques can be employed to establish well-posedness. For nonlinear nonlocal systems, additional assumptions concerning continuity, monotonicity, compactness, and nonlinear growth conditions may be necessary. The uniqueness of solutions is particularly important in applications involving predictive simulations because multiple mathematical solutions may produce inconsistent physical interpretations.

4.2 Regularity of Solutions

Solution regularity concerns the smoothness and differentiability of the mathematical functions satisfying a nonlocal equation. Unlike classical elliptic PDEs, where regularity is frequently investigated through local differential operators, nonlocal equations require analysis of integral interactions across extended spatial domains. Fractional elliptic equations frequently exhibit different regularity properties from classical second-order elliptic equations. In particular, the behaviour of solutions near domain boundaries can be strongly influenced by the nonlocal operator and its exterior data.

4.3 Stability and Energy Estimates

Stability analysis determines whether small perturbations in initial conditions or external inputs lead to controlled changes in the solution. For symmetric nonlocal diffusion operators, energy estimates can be derived using suitable nonlocal energy functionals. A representative energy expression is

$$E(u) = \frac{1}{4}\int_{\Omega}\int_{\Omega} K(x,y)[u(y) - u(x)]^2\, dy\, dx.$$

When the kernel is nonnegative and symmetric, this expression is nonnegative and provides a mathematical measure of the interaction energy. Energy-based approaches are widely employed to establish stability, convergence, and dissipative properties of nonlocal systems.

4.4 Boundary and Exterior Conditions

Boundary conditions in nonlocal PDEs differ significantly from those used in classical differential equations. For a conventional local PDE, boundary values are commonly prescribed directly on the boundary of the computational domain. In a nonlocal model, however, points outside the domain may interact with points inside it. Consequently, appropriate exterior data may need to be prescribed over an extended interaction region. For a domain Ω, a nonlocal interaction model may require information over

$$\Omega_\delta = \{y \in \mathbb{R}^n : \mathrm{dist}(y, \Omega) < \delta\}.$$

The treatment of exterior conditions is therefore essential for obtaining physically meaningful and mathematically consistent solutions.

5. Numerical Approximation Techniques for Nonlocal PDEs

Numerical approximation is essential for solving nonlocal equations because analytical solutions are available only for a limited number of mathematical configurations. The integral structure of nonlocal operators introduces additional computational complexity compared with conventional local differential equations.

5.1 Finite Difference Methods

Finite difference methods approximate derivatives and integral interactions through discrete spatial and temporal grids. For a one-dimensional nonlocal diffusion equation,

$$\frac{\partial u(x,t)}{\partial t} = \int_{\Omega} K(x,y)[u(y,t) - u(x,t)]\, dy,$$

the integral term may be approximated using numerical quadrature:

$$\mathcal{L}_{NL}u(x_i) \approx \sum_{j=1}^{N} w_j\, K(x_i, x_j)(u_j - u_i),$$

where w_j represents the quadrature weight associated with spatial point x_j. The resulting discrete system can be expressed in matrix form as

$$\frac{d\mathbf{u}}{dt} = A\mathbf{u} + \mathbf{f},$$

where A is a matrix representing nonlocal interactions. Finite difference approaches are relatively straightforward to implement and can provide accurate solutions for regular computational domains. However, their computational complexity increases when every spatial point interacts with a large number of other points.

5.2 Finite Element Methods

Finite element methods are particularly useful for nonlocal problems involving complex geometries, heterogeneous materials, and irregular spatial domains. The solution is approximated through a finite-dimensional basis:

$$u_h(x,t) = \sum_{i=1}^{N} U_i(t)\phi_i(x),$$

where $\phi_i(x)$ represents a basis function and $U_i(t)$ is the corresponding coefficient. The weak formulation of a nonlocal diffusion problem can involve the bilinear form

$$a(u,v) = \frac{1}{2}\int_{\Omega}\int_{\Omega} K(x,y)[u(y)-u(x)][v(y) - v(x)]\,dy\,dx.$$

Finite element methods provide a flexible framework for investigating nonlocal operators with variable coefficients and complex domain structures. However, the resulting stiffness matrices are often dense because the interaction between distant spatial points is not necessarily zero.

5.3 Spectral Methods

Spectral methods approximate unknown solutions using global basis functions, such as Fourier series, Chebyshev polynomials, or Legendre polynomials. A general spectral approximation can be expressed as

$$u_N(x,t) = \sum_{k=0}^{N} a_k(t)\psi_k(x),$$

where $\psi_k(x)$ represents a selected basis function.
These methods can achieve high accuracy when solutions are sufficiently smooth and the computational domain has suitable geometric properties. Fourier-based spectral approaches are especially useful for translation-invariant nonlocal operators because convolution structures can be evaluated efficiently in the frequency domain.

5.4 Discontinuous Galerkin Methods

Discontinuous Galerkin (DG) methods employ piecewise polynomial approximations that do not necessarily maintain continuity across neighbouring computational elements. These methods are attractive for nonlocal transport problems involving sharp gradients, discontinuities, and heterogeneous coefficients. Their flexibility allows local mesh refinement and the development of high-order numerical approximations. Nevertheless, nonlocal interactions require communication between computational elements beyond immediate neighbours, which can increase implementation complexity and computational costs.

5.5 Fast Convolution and Fourier-Based Algorithms

A major computational challenge in nonlocal models is the evaluation of integral interactions.
For translation-invariant kernels, the integral operator may be represented as a convolution:

$$\mathcal{L}_{NL}u = K * u - u\int K(z)\,dz.$$

The convolution term can be evaluated using Fourier transforms:

$$\mathcal{F}(K*u) = \widehat{K}(\xi)\hat{u}(\xi).$$

Fast Fourier transform algorithms can significantly reduce computational costs when the domain and kernel structure satisfy the necessary assumptions. Other acceleration techniques include hierarchical matrices, fast multipole methods, low-rank approximations, and sparse kernel representations.

6. Applications of Nonlocal PDEs in Multiscale Systems

6.1 Anomalous Diffusion and Transport Phenomena

Anomalous diffusion is one of the most important application areas of nonlocal mathematical modelling. In classical diffusion, the mean-square displacement of a particle generally follows

$$\langle r^2(t)\rangle \propto t.$$

However, anomalous transport may follow

$$\langle r^2(t)\rangle \propto t^{\alpha},$$

where $\alpha \neq 1$.
When $0 < \alpha < 1$, the process is commonly associated with subdiffusion, whereas $\alpha > 1$ may indicate superdiffusive behaviour. Fractional diffusion equations provide mathematical representations of such processes:

$$\frac{\partial u}{\partial t} = -D_{\alpha}(-\Delta)^{\alpha/2}u.$$

These formulations are relevant to particle transport in heterogeneous porous media, biological tissues, complex fluids, and disordered materials.

6.2 Fluid Mechanics and Turbulent Transport

Nonlocal mathematical formulations have potential applications in fluid mechanics, particularly in modelling anomalous transport, turbulent dispersion, and systems involving long-range interactions. In conventional fluid mechanics, the Navier–Stokes equations describe the conservation of mass and momentum through local differential operators. Nonlocal modifications can incorporate spatially extended transport mechanisms or fractional diffusion terms:

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u}\cdot\nabla)\mathbf{u} = -\frac{1}{\rho}\nabla p - \nu(-\Delta)^{s}\mathbf{u} + \mathbf{f}.$$

Here, the fractional diffusion term provides an alternative mathematical representation of momentum dissipation. Such formulations are investigated in the mathematical analysis of complex flows, turbulence models, and anomalous fluid transport.

6.3 Material Science and Peridynamic Modelling

Peridynamics represents an important development in nonlocal continuum mechanics. Classical elasticity relies on spatial derivatives of displacement fields to calculate strain and stress. These formulations may become difficult to apply directly when cracks, fractures, or discontinuities are present. Peridynamics replaces the classical differential formulation with integral interactions between material points. A simplified peridynamic equation of motion can be expressed as

$$\rho(x)\frac{\partial^2 u(x,t)}{\partial t^2} = \int_{H_x} f\,(u(y,t)-u(x,t), y-x)\,dy + b(x,t),$$

where H_x represents the interaction neighbourhood, f is the force density function, and b denotes the external body force. This formulation allows displacement discontinuities to develop naturally without requiring conventional spatial derivatives at crack surfaces. Important applications include fracture mechanics, composite materials, fatigue analysis, impact damage, and structural failure prediction.

6.4 Biological Systems and Population Dynamics

Biological systems frequently involve interactions across multiple spatial scales. Organisms may respond to nearby environmental conditions while simultaneously exhibiting long-distance dispersal, migration, or communication. A nonlocal reaction-diffusion equation can be represented as

$$\frac{\partial u(x,t)}{\partial t} = \int_{\Omega} K(x,y)[u(y,t) - u(x,t)]\,dy + R(u),$$

where $u(x,t)$ represents population density and $R(u)$ describes local biological growth or interaction. Nonlocal models are particularly useful for investigating:

- Population dispersal and species migration.
- Ecological invasion and spatial competition.
- Biological pattern formation.
- Cell migration and tissue development.
- Spatially distributed disease transmission.

These applications demonstrate how nonlocal formulations can represent biological interactions extending beyond immediate geographical neighbourhoods.

6.5 Environmental Modelling and Groundwater Transport

Environmental systems often exhibit heterogeneous transport properties because of variations in soil structure, porosity, permeability, and geological composition. Conventional diffusion models may not adequately represent preferential flow, long-distance contaminant movement, or anomalous transport in fractured geological formations. Nonlocal transport equations provide an alternative approach:

$$\frac{\partial C(x,t)}{\partial t} = \int_{\Omega} K(x,y)[C(y,t) - C(x,t)]\,dy + S(x,t),$$

where $C(x,t)$ represents contaminant concentration and $S(x,t)$ denotes external sources or sinks. Such formulations may be useful in studying groundwater contamination, pollutant migration, soil-water interactions, and transport through heterogeneous geological structures.

6.6 Image Processing and Signal Analysis

Nonlocal operators have also become important in image processing and computer vision. Unlike local image-processing methods, which primarily use neighbouring pixels, nonlocal approaches identify similarities between spatially separated image regions. The nonlocal means algorithm is a prominent example. It estimates the intensity of a pixel through a weighted average of other pixels exhibiting similar neighbourhood patterns. A simplified mathematical representation is

$$u(x) = \frac{\int_{\Omega} w(x,y)v(y)\,dy}{\int_{\Omega} w(x,y)\,dy},$$

where $v(y)$ represents the original noisy image and $w(x,y)$ denotes a similarity-based weight. These approaches are useful for image denoising, texture preservation, medical image enhancement, and feature reconstruction.

6.7 Complex Networks and Long-Range Interactions

Complex systems frequently exhibit interactions that cannot be adequately represented through nearest-neighbour relationships. Examples include communication networks, transportation systems, social networks, and interconnected infrastructure. Nonlocal mathematical models can incorporate long-distance coupling and distributed interactions through integral operators. Their application provides opportunities for investigating synchronization, network stability, information propagation, and collective behaviour in complex systems.

7. Nonlocal PDEs in Multiscale Mathematical Modelling

Multiscale modelling is concerned with mathematical systems in which different spatial or temporal scales contribute to the overall behaviour of a process. Nonlocal PDEs provide a natural framework for representing interactions across these scales because their operators incorporate information from spatially extended regions.

7.1 Spatial Multiscale Interactions

Spatial multiscale systems involve processes operating at microscopic, mesoscopic, and macroscopic levels. For example, the transport of contaminants through porous materials may depend on microscopic pore structures, intermediate-scale flow channels, and macroscopic groundwater movement. A nonlocal interaction kernel can incorporate these spatial dependencies:

$$\mathcal{L}_{NL}u(x) = \int_{\Omega} K_{\delta}(x,y)[u(y) - u(x)]\,dy.$$

The interaction horizon δ can be adjusted to represent different characteristic spatial scales.

7.2 Temporal Multiscale Behaviour

Temporal multiscale systems involve processes with different characteristic relaxation or response times. Fractional derivatives and memory kernels provide a mathematical mechanism for representing such behaviour. A time-fractional equation can be expressed as

$${}^{C}D_t^{\alpha}u(x,t) = D\Delta u(x,t) + f(x,t),$$

where ${}^{C}D_t^{\alpha}$ denotes the Caputo fractional derivative. For $0 < \alpha < 1$, this operator incorporates information from the previous temporal history of the system.

7.3 Coupled Local and Nonlocal Systems

Many real-world systems involve simultaneous local and nonlocal interactions. A combined mathematical model can be written as

$$\frac{\partial u}{\partial t} = D\Delta u + \lambda \int_{\Omega} K(x,y)[u(y,t) - u(x,t)]\, dy + R(u),$$

where D represents the local diffusion coefficient and λ controls the relative strength of nonlocal interactions. Such formulations provide a flexible approach for modelling systems in which short-range transport and long-range interactions coexist.

7.4 Multiscale Model Reduction

Model reduction aims to simplify complex mathematical systems while preserving their essential behaviour. In nonlocal modelling, this may involve approximating dense interaction operators through low-rank representations, reduced-order basis functions, or simplified interaction kernels.

These approaches are particularly important for high-dimensional simulations, where direct evaluation of all nonlocal interactions may become computationally prohibitive.

Table 1: Applications of nonlocal PDEs across different scales

Research area	Dominant scale	Nonlocal mechanism	Potential application
Porous media	Micro–macro	Long-range transport	Groundwater contamination
Biological systems	Cellular–organismal	Spatial dispersal	Population dynamics
Material science	Atomic–structural	Long-range forces	Fracture prediction
Fluid mechanics	Mesoscopic–macroscopic	Fractional dissipation	Complex flow modelling
Environmental systems	Local–regional	Nonlocal transport	Pollutant migration
Image processing	Pixel–image	Similarity-based interactions	Noise reduction
Network science	Node–network	Long-range coupling	Synchronisation

8. Emerging Computational Approaches and Artificial Intelligence

The increasing dimensionality and complexity of nonlocal mathematical models have encouraged the development of advanced computational methods.

8.1 Physics-Informed Neural Networks

Physics-informed neural networks (PINNs) combine neural network approximations with governing mathematical equations. A neural network is trained to approximate the solution $u(x,t)$ while minimising a loss function incorporating the governing PDE, initial conditions, and boundary or exterior conditions.

For a nonlocal equation, the residual may be expressed as

$$\mathcal{R}(x,t) = \frac{\partial u_\theta}{\partial t} - \int_{\Omega} K(x,y)[u_\theta(y,t) - u_\theta(x,t)]\, dy - f(x,t),$$

where u_θ denotes the neural network approximation. The corresponding loss function may include

$$\mathcal{L} = \mathcal{L}_{PDE} + \lambda_1 \mathcal{L}_{IC} + \lambda_2 \mathcal{L}_{BC},$$

where the terms represent the PDE residual, initial-condition error, and boundary or exterior-condition error. A major difficulty is the efficient evaluation of the nonlocal integral during training. This may require numerical quadrature, sampling-based approximations, or specialised neural architectures.

8.2 Deep Learning-Based Operator Approximation

Deep learning can also be used to approximate mappings between input functions and solution functions. Neural operators, including Fourier neural operators and related architectures, have attracted attention for solving parameterised differential equations. For nonlocal problems, such methods may learn approximate relationships between interaction kernels, initial conditions, and resulting solutions. However, their reliability depends on training-data coverage, operator generalisation, numerical stability, and physical consistency.

8.3 Hybrid Numerical and Machine Learning Methods

Hybrid approaches combine conventional numerical discretisation with machine learning-based approximation. For example, a finite element solver may be used to construct accurate training data, while a neural network approximates expensive integral interactions. This strategy may reduce computational costs in repeated simulations, parameter estimation, and inverse modelling. Nevertheless, numerical verification remains essential because a low training error does not automatically establish mathematical accuracy or physical validity.

8.4 Data-Driven Identification of Nonlocal Kernels

Another emerging research direction involves identifying unknown interaction kernels from experimental or simulation data. A general inverse problem can be formulated as

$$\min_K \|u_{\text{observed}} - u_{\text{model}}(K)\|^2 + \lambda \mathcal{R}(K),$$

where K represents the unknown kernel and $\mathcal{R}(K)$ is a regularisation term.

This approach may be useful in identifying effective interaction mechanisms in biological systems, heterogeneous materials, and environmental transport processes.

11. Conclusion

Nonlocal partial differential equations represent an important extension of classical mathematical modelling, providing a versatile framework for describing systems involving long-range interactions, spatial heterogeneity, anomalous diffusion, and

temporal memory. Their mathematical formulations, based on fractional derivatives, integral operators, convolution kernels, and nonlocal interaction functions, offer capabilities that are not readily available through conventional local differential equations. The theoretical analysis of these models involves important considerations concerning existence, uniqueness, regularity, stability, and boundary or exterior conditions. Numerical methods, including finite difference, finite element, spectral, discontinuous Galerkin, and fast convolution techniques, provide essential computational tools for approximating solutions of nonlocal systems. However, their effectiveness depends on the mathematical structure of the interaction kernel, computational domain, and complexity of the underlying physical process.

The applications of nonlocal PDEs extend across material science, fluid mechanics, biological systems, environmental transport, image processing, and complex networks. Their ability to represent spatially extended dependencies makes them particularly valuable for multiscale systems in which microscopic interactions influence macroscopic behaviour. Recent developments in machine learning, neural operators, physics-informed neural networks, and data-driven kernel identification have created additional opportunities for improving the computational efficiency of nonlocal models. Nevertheless, challenges associated with computational complexity, kernel selection, numerical convergence, parameter uncertainty, and experimental validation remain important research concerns. Future progress will require closer integration of mathematical analysis, numerical computation, and application-specific validation.

Overall, nonlocal partial differential equations provide a powerful and expanding mathematical framework for investigating complex systems whose behaviour extends beyond the assumptions of traditional local models.

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