Discover Engineering An International Journal
OPEN ACCESS
ISSN 3117-566X
Original Research Article

Classification of Essentially Algebraic Composition Operators on Weighted Lorentz–Karamata Sequence Spaces

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Citation: Ajay Kumar Sharma, and Malkesh Singh (2025). Classification of Essentially Algebraic Composition Operators on Weighted Lorentz–Karamata Sequence Spaces. *Discover Engineering: An International Journal.*
DOI: https://doi.org/10.51470/DE.2025.6.1.17

Received 15 January 2025 | Revised 16 February 2025 | Accepted 17 March 2025 | Available Online 17 April 2025

Abstract

We characterize essentially algebraic composition operators on weighted Lorentz–Karamata sequence spaces. Under natural hypotheses—finite-to-one self-map, uniformly comparable weights on periodic cycles, and regular variation of the slowly varying function—we establish the tree-selection lemma, the enter lemma, and derive the compactness criterion from regular-variation estimates for the fundamental function. The slowly varying factor does not change the zero/nonzero nature of the asymptotic ratios that determine compactness and essential algebraicity.

Keywords: *Composition operators; Algebraic operators; Essentially algebraic operators; Lorentz–Karamata spaces.*
2020 Mathematics Subject Classification: *Primary 47B33; Secondary 46E30, 47B07.*

1 Introduction

Let $X = (X, \Sigma, \mu)$ be a σ-finite complete measure space and let $T: X \to X$ be a measurable transformation, that is, $T^{-1}(A) \in \Sigma$ for any $A \in \Sigma$. If $\mu \circ T^{-1}(A) = 0$ for each $A \in \Sigma$ with $\mu(A) = 0$, then T is said to be non-singular.

Any non-singular measurable transformation T induces a linear operator C_T from $L^0(X)$ into itself defined by
$(C_T f)(t) = f \circ T(t) = f(T(t)), \quad t \in X, \quad f \in L^0(X),$ (1)

where $L^0(X)$ denotes the linear space of all equivalence classes of Σ-measurable functions on X. Here we identify any two functions that are equal μ-almost everywhere on X.

Let M_0 be the class of all functions f in $L^0(X)$ that are finite μ-almost everywhere on X. For $f \in M_0$, we define the distribution function μ_f of f on $(0, \infty)$ by
$\mu_f(\lambda) = \mu(\{x \in X: |f(x)| > \lambda\})$

and the decreasing rearrangement f^* of f on $(0, \infty)$ by
$f^*(t) = \inf\{\lambda > 0: \mu_f(\lambda) \le t\} = \sup\{\lambda > 0: \mu_f(\lambda) > t\}.$

The Lorentz space $L^{p,q}(\mu)$ is the set of all classes of Σ-measurable functions f on X such that the functional $\| f \|_{pq} < \infty$, where

$$\| f \|_{pq} = \begin{cases} \left(\int_0^\infty (t^{1/p} f^*(t))^q \dfrac{dt}{t} \right)^{1/q}, & \text{if} 1 < p < \infty, 1 \le q < \infty, \\ \sup\limits_{t>0} t^{1/p} f^*(t), & \text{if} 1 < p \le \infty, q = \infty. \end{cases}$$

Take $X = \mathbb{N}, \Sigma = 2^{\mathbb{N}}$ and $\mu(\{n\}) = 1$. Then the Lorentz sequence space $\ell^{p,q}$ is the set of all sequences $a = \{a_n\} \in c_0$ such that the functional $\| a \|_{pq} < \infty$, where

$$\| a \|_{pq} = \begin{cases} \left(\sum\limits_{n=1}^\infty (n^{1/p} a_n^*)^q n^{-1} \right)^{1/q}, & \text{if} 1 < p < \infty, 1 \le q < \infty, \\ \sup\limits_{n\ge1} n^{1/p} a_n^*, & \text{if} 1 < p \le \infty, q = \infty. \end{cases}$$

If we take $X = \mathbb{N}, \Sigma = 2^{\mathbb{N}}$ and $\mu: \mathbb{N} \to (0, \infty)$ is the weight function, then the corresponding Lorentz sequence space with weight μ is denoted by $\ell_\mu^{p,q}$ and

$$\| a \|_{pq,\mu} = \begin{cases} \left(\sum\limits_{n=1}^\infty (n^{1/p} a_n^* \mu(n))^q \right)^{1/q}, & \text{if} 1 < p < \infty, 1 \le q < \infty, \\ \sup\limits_{n\ge1} n^{1/p} a_n^* \mu(n), & \text{if} 1 < p \le \infty, q = \infty. \end{cases}$$

Note that the Lorentz spaces are quasi-normed linear spaces and the functional $\|\cdot\|_{pq}$ is a norm if and only if $1 \le q \le p < \infty$ or $p = q = \infty$. For any Σ-measurable set A of finite measure, we have

$$\| \chi_A \|_{pq} = \begin{cases} (p/q)^{1/q} (\mu(A))^{1/p}, & \text{if} 1 < p < \infty, 1 \le q < \infty, \\ (\mu(A))^{1/p}, & \text{if} 1 < p \le \infty, q = \infty. \end{cases}$$

For details about Lorentz spaces one can refer to [1, 5] and references therein.

Theorem 1.1 ([6, Theorem 2.3]). *Let $T: X \to X$ be a non-singular measurable transformation. Then T induces a bounded composition operator C_T on $L^{pq}(\mu)$, $1 < p < \infty$, $1 \le q \le \infty$ if and only if there exists some constant $M > 0$ such that $\mu \circ T^{-1}(A) \le M\mu(A)$,*

for each $A \in \Sigma$. Moreover, $\| C_T \| = \sup\limits_{\substack{A\in\Sigma \\ 0<\mu(A)<\infty}} \left(\frac{\mu\circ T^{-1}(A)}{\mu(A)} \right)^{1/p}$.

For Orlicz spaces see [9, 13, 14] and for composition operators on Orlicz spaces one can refer to [8, 14] and references therein. The work of this paper is motivated by the interesting work of Böttcher and Heidler [3, 4].

Definition 1.2 Let $\mathbb{C}[z]$ denote the ring of univariate polynomials with complex coefficients. A polynomial
$f(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0 \in \mathbb{C}[z]$
is said to be monic if $a_n = 1$.

Definition 1.3 An operator U on a Banach space B is said to be algebraic if there is a non-zero polynomial $p(z)$ such that $p(U) = 0$ and U will be called essentially algebraic if there is a non-zero polynomial $q(z)$ such that $q(U)$ is compact.

Definition 1.4 The monic polynomial $p(z)$ of the least degree Such that $p(U)$ is zero is called the characteristic polynomial of U.

Definition 1.5 The monic polynomial $q(z)$ of the least degree such that $q(U)$ is compact is called the essentially characteristic polynomial of U.

Let p_a and q_a represent the characteristic and essentially characteristic polynomials associated with the linear operator U. Also for a polynomial $p(z) \in \mathbb{C}[z]$, $(p(z))$ denotes the two sided ideal $p(z)\mathbb{C}[z]$.

Let $B(X)$ be the collection of all bounded operators on a Banach space X and $K(X)$ be the collection of all compact operators on X. $B(X)$ is a Banach algebra under the operator norm and $K(X)$ is a two sided ideal in $B(X)$. Let $\pi: B(X) \to B(X)/K(X)$ be the natural map of $B(X)$ onto the Calkin algebra $B(X)/K(X)$. Let $\mathrm{alg}(U) = \{p(U): p(z) \in \mathbb{C}[z]\}$ and $\mathrm{alg}_\pi(U) = \{p(\pi(U)): p(z) \in \mathbb{C}[z]\}$. In case $\mathrm{alg}(U)$ is finite dimensional, $\mathrm{alg}(U)$ is isomorphic to $\mathbb{C}[z]/(p_a(z))$ and is a closed subalgebra of $B(X)$. In case $\mathrm{alg}_\pi(U)$ is finite dimensional, $\mathrm{alg}_\pi(U)$ is isomorphic to $\mathbb{C}[z]/(q_a(z))$ and is a closed subalgebra of $B(X)/K(X)$. The essential characteristic polynomial of U is actually the characteristic polynomial of the Calkin image $\pi(U)$ of U in the Calkin algebra $B(X)/K(X)$. See [3, 4] for more details about the above definitions and notations.

Algebraic and essentially algebraic composition operators on $\ell^p, 1 \le p \le \infty$ are studied by Böttcher and Heidler [4] and these results were extended to Orlicz spaces by Kumar and Kumar [5].
2 Weighted Lorentz–Karamata Sequence Spaces

Let $\Omega = \mathbb{N}, \Sigma = 2^{\mathbb{N}}$, and let $\mu_n > 0$ be a weight sequence. Define the weighted counting measure

$\mu(A) = \sum_{n\in A} \mu_n, \qquad A \subseteq \mathbb{N}$. (2)

Let $T: \mathbb{N} \to \mathbb{N}$ be a self-map. Since $\mu_n > 0$, every subset of measure zero is empty, so every self-map is nonsingular. The composition operator is
$(C_T f)(n) = f(T(n))$.

Let $b: (0,\infty) \to (0,\infty)$ be a slowly varying function. For $1 < p < \infty, \qquad 1 \le q < \infty,$

the **weighted Lorentz–Karamata sequence space** $LK^{p,q}_{b,\mu}$ is the set of all sequences $a = \{a_n\}$ such that

$\| a \|_{p,q,b,\mu} = \left(\int_0^\infty \left(t^{1/p} b(t) a^*(t)\right)^q \frac{dt}{t}\right)^{1/q} < \infty$, (3)

where a^* is the decreasing rearrangement of a with respect to μ. When $b \equiv 1$, $LK^{p,q}_{b,\mu}$ is the usual weighted Lorentz sequence space $L^{p,q}(\mu)$.

Assumption 2.1 The function $b: (0,\infty) \to (0,\infty)$ is slowly varying at 0 and at ∞, locally bounded, and positive. We write $b \in SV_0 \cap SV_\infty$. Under Assumption 2.1, the space $LK^{p,q}_{b,\mu}$ with the functional $\|\cdot\|_{p,q,b,\mu}$ is a rearrangement-invariant Banach function space with the Fatou property (see [11, 12, 10] for details). Throughout, $LK^{p,q}_{b,\mu}$ denotes this Banach realization.

Assumption 2.2 The self-map $T: \mathbb{N} \to \mathbb{N}$ is finite-to-one: for every $n \in \mathbb{N}$, the fiber $T^{-1}(\{n\})$ is finite.

Assumption 2.3 There is a constant $K \ge 1$ such that for every periodic cycle $\{n, T(n), \ldots, T^{s-1}(n)\}$ of T, and for all $r, r' \in \{0, \ldots, s-1\}$,

$$K^{-1} \le \frac{\mu(\{T^r(n)\})}{\mu(\{T^{r'}(n)\})} \le K.$$

Definition 2.4 For $1 < p < \infty, 1 \le q < \infty$, define the **fundamental function** analytically by

$$\Phi(s) = \left(\int_0^s \left(t^{1/p} b(t)\right)^q \frac{dt}{t}\right)^{1/q}, \qquad s > 0.$$

Lemma 2.5 *For every measurable set A of finite measure,* $\| \chi_A \|_{p,q,b,\mu} = \Phi(\mu(A))$.
Proof. The distribution function of χ_A is $\mu_{\chi_A}(\lambda) = \mu(A)$ for $0 < \lambda < 1$ and 0 for $\lambda \ge 1$. Hence $(\chi_A)^*(t) = \chi_{[0,\mu(A))}(t)$. Substitution in the norm gives the result.

Proposition 2.6 *Under Assumption 2.1, for* $q < \infty, \Phi(s) \sim \left(\frac{p}{q}\right)^{1/q} s^{1/p} b(s) \qquad (s \to 0 \text{ors} \to \infty)$.
In particular, $\Phi \in RV_{1/p}$ *at both endpoints.*

Proof. Write the integrand as $t^{q/p-1} b(t)^q$. Since b^q is slowly varying and $q/p > 0$, Karamata's theorem (see [2, Theorem 1.5.11]) gives

$$\int_0^s t^{q/p-1}\, b(t)^q\, dt \sim \frac{p}{q} s^{q/p} b(s)^q \qquad (s \to 0 \text{ors} \to \infty).$$

Taking q-th roots yields the stated asymptotic. □

Corollary 2.7 *Under Assumption 6, Φ satisfies the doubling condition* $1 < c_0 \le \frac{\Phi(2s)}{\Phi(s)} \le C_0 < \infty \qquad (s > 0)$.
Proof. By Proposition 2.6, the ratio $\Phi(2s)/\Phi(s)$ tends to $2^{1/p} > 1$ as $s \to 0$ and as $s \to \infty$. Since Φ is continuous and positive on every compact subinterval $[a, b] \subset (0,\infty)$, the ratio is bounded above and below by positive constants there. Combining the two endpoint limits with the compact-interval bound yields global constants c_0, C_0 with $1 < c_0 \le C_0 < \infty$. □

Lemma 2.8 (Global scaling and inversion estimates).
Under Assumption 2.1, *the following hold.*

1. ***(Scaling estimate.)*** *For every* $\delta \in (0, 1/p)$ *there exists* $C_\delta \ge 1$ *such that for all* $0 < \lambda \le 1$ *and all* $s > 0$, $\Phi(\lambda s) \le C_\delta\, \lambda^{1/p-\delta}\, \Phi(s)$. *Equivalently, with* $\alpha = 1/p - \delta > 0$, $\Phi(\lambda s) \le C_\alpha\, \lambda^\alpha\, \Phi(s)\ \ (0 < \lambda \le 1, s > 0)$.

2. ***(Inversion estimate.)*** *For every* $\eta > 0$ *there exists* $\varepsilon > 0$ *such that for all* $s, t > 0$, $\Phi(s) < \varepsilon\, \Phi(t) \Longrightarrow s < \eta\, t$.

Proof. Since $\Phi \in RV_{1/p}$ at both endpoints and is continuous and positive on $(0,\infty)$, it satisfies Potter-type global bounds: for every $\delta > 0$ there is a constant $C_\delta \ge 1$ such that for all $s, t > 0$,

$$\frac{1}{C_\delta}\left(\frac{s}{t}\right)^{1/p-\delta} \le \frac{\Phi(s)}{\Phi(t)} \le C_\delta \left(\frac{s}{t}\right)^{1/p+\delta} \qquad (s \le t),$$

and the reverse inequalities for $s \ge t$. These bounds follow from the Uniform Convergence Theorem for regularly varying functions together with the continuity and positivity of Φ on compact subsets of $(0,\infty)$ (see [2, Theorem1.5.6] and [1, Chapter 2]).

(i) Fix $\delta \in (0, 1/p)$ and let $\lambda \in (0,1]$. Applying the upper Potter bound with $s = \lambda t$ and $t = t$ (so $s \le t$), we get

$$\frac{\Phi(\lambda t)}{\Phi(t)} \le C_\delta\, \lambda^{1/p-\delta}.$$

Since this holds for every $t > 0$, we obtain the stated estimate with $\alpha = 1/p - \delta$.

(ii) Let $\eta > 0$ be given. Choose $\delta \in (0, 1/p)$ and let $C_\delta \ge 1$ be the constant from the global Potter bound. Suppose $\Phi(s) < \varepsilon\Phi(t)$ for some $\varepsilon > 0$. If $s \ge t$, then $\Phi(s) \ge \Phi(t)$ (since Φ is increasing), so $\varepsilon > 1$ and $s < \eta t$ is trivial for $\eta \ge 1$; if $0 < \eta < 1$, we may assume $s < t$ without loss of generality, since otherwise the conclusion fails only for the trivial case which we exclude by choosing $\varepsilon < 1$.

Assume therefore $s < t$. Set $\lambda = s/t \in (0,1)$. Applying the lower Potter bound with $s = \lambda t, t = t$, we get

$$\frac{\Phi(\lambda t)}{\Phi(t)} \ge \frac{1}{C_\delta}\, \lambda^{1/p+\delta}.$$

Combining with $\Phi(s) < \varepsilon\Phi(t)$,

$$\frac{1}{C_\delta}\, \lambda^{1/p+\delta} \le \frac{\Phi(s)}{\Phi(t)} < \varepsilon,$$

hence
$\lambda < (C_\delta \varepsilon)^{1/(1/p+\delta)}$.

Set
$\eta(\varepsilon) = (C_\delta \varepsilon)^{1/(1/p+\delta)}$.

Then $\eta(\varepsilon) \to 0$ as $\varepsilon \to 0$, and $s = \lambda t < \eta(\varepsilon)t$.
This proves (ii). □

Theorem 2.9 (Boundedness criterion). *Let $T: \mathbb{N} \to \mathbb{N}$ be a self-map. Then T induces a bounded composition operator $C_T \in \mathcal{B}(LK^{p,q}_{b,\mu})$ if and only if $\sup_{A\subset\mathbb{N}} \frac{\Phi(\mu(T^{-1}(A)))}{\Phi(\mu(A))} < \infty$, where the supremum is taken over all A with $0 < \mu(A) < \infty$. Moreover, the operator norm is equivalent to this supremum.*

Proof. Since $LK^{p,q}_{b,\mu}$ is a rearrangement-invariant Banach function space (see [11, 12]), the composition operator C_T is bounded if and only if the measure $\mu \circ T^{-1}$ is absolutely continuous with respect to μ with a bounded Radon–Nikodym derivative in the associate norm. Concretely: Suppose first that there is $M > 0$ such that $\mu(T^{-1}(A)) \le M\mu(A)$ for every A of finite measure.
For $f \in LK^{p,q}_{b,\mu}$ and $\lambda > 0$,

$$\mu(\{|C_T f| > \lambda\}) = \mu(T^{-1}(\{|f| > \lambda\})) \le M\mu(\{|f| > \lambda\}).$$

Hence $(C_T f)^*(t) \le f^*(t/M)$ for all $t > 0$. Therefore

$$\| C_T f \|^q_{p,q,b,\mu} = \int_0^\infty \left(t^{1/p} b(t)(C_T f)^*(t)\right)^q \frac{dt}{t} \le \int_0^\infty \left(t^{1/p} b(t) f^*(t/M)\right)^q \frac{dt}{t}.$$

Substituting $u = t/M$,

$$= M^{q/p} \int_0^\infty \left(u^{1/p} b(Mu) f^*(u)\right)^q \frac{du}{u}.$$

By the doubling property of Φ (equivalently, the slow variation of b), there is a constant C_M depending only on M such that $b(Mu) \le C_M b(u)$ for all $u > 0$.
Hence
$\| C_T f \|_{p,q,b,\mu} \le C_M^{1/q} M^{1/p} \| f \|_{p,q,b,\mu}$,
so C_T is bounded.

Conversely, suppose C_T is bounded. For every A of finite positive measure, $\chi_A \in LK^{p,q}_{b,\mu}$ and

$$\| C_T \chi_A \|_{p,q,b,\mu} = \| \chi_{T^{-1}(A)} \|_{p,q,b,\mu} = \Phi(\mu(T^{-1}(A))),$$

while $\| \chi_A \|_{p,q,b,\mu} = \Phi(\mu(A))$. Hence

$$\Phi(\mu(T^{-1}(A))) \le \| C_T \| \ \Phi(\mu(A)),$$

which gives the stated supremum bound. □

3 Essential Orbit Structure

For a self-map $T: \mathbb{N} \to \mathbb{N}$, the set $\{n, T(n), T^2(n), \dots\}$ is called the orbit of n. If the orbit of $n \in \mathbb{N}$ is a finite set, then we define the enter length $\mathrm{ent}_T(n)$ and the cycle length $\mathrm{cyc}_T(n)$ of the point n as
$\mathrm{ent}_T(n) = \min\{k \in \mathbb{N} \cup \{0\}: T^k(n) = T^{k+m}(n) \text{for some} m \ge 1\}$
and
$\mathrm{cyc}_T(n) = \min\{k \in \mathbb{N} \cup \{0\}: T^{k+\mathrm{ent}_T(n)}(n) = T^{\mathrm{ent}_T(n)}(n)\}$.
If the orbit is infinite, then we take $\mathrm{ent}_T(n) = \mathrm{cyc}_T(n) = \infty$.
The maximal enter length of the self-map $T: \mathbb{N} \to \mathbb{N}$ is denoted by $\mathrm{ent}(T)$ and defined by $\mathrm{ent}(T) = \sup_{n\in\mathbb{N}} \mathrm{ent}_T(n)$

and the set of occurring periods or cycle lengths is
$\mathrm{Per}(T) = \{k \in \mathbb{N}: N_k(T) \ne \phi\}$,
where
$N_k(T) = \{n \in \mathbb{N}: \mathrm{ent}_T(n) = 0 \text{and} \mathrm{cyc}_T(n) = k\}, \quad (k \ge 1)$.

In case $|N_k(T)| = \mathrm{card}(N_k(T)) = \infty$, we say that k is an essential period of T. The set of all essential periods is denoted by $W\mathrm{per}(T)$. The set of all unessential periods is defined as
$U\mathrm{per}(T) = \mathrm{Per}(T) \backslash W\mathrm{per}(T)$
and the set of all periodic points with unessential periods is the set
$N_U(T) = \{n \in \mathbb{N}: \mathrm{ent}_T(n) = 0 \text{ and } \mathrm{cyc}_T(n) \in U\mathrm{per}(T)\}$.
The essential enter length $W\mathrm{ent}_\Phi(T)$ for the self-map $T: \mathbb{N} \to \mathbb{N}$ depends on the decreasing rearrangement $*$ and on μ. If the set
$N_0(T) = \{n \in \mathbb{N}: \mathrm{ent}_T(n) \ge 1\}$
is finite, then we put $W\mathrm{ent}_\Phi(T) = 0$. If $N_0(T)$ is infinite, then we define $W\mathrm{ent}_\Phi(T)$ as the minimal $m \in \mathbb{N}$ such that for every $\varepsilon > 0$ there is a finite set $F \subset N_0(T)$ with

$$\sup_{n\in N_0(T)\backslash F} \frac{\Phi(\mu(T^{-m}(\{n\})))}{\Phi(\mu(\{n\}))} < \varepsilon. \ (4)$$

Equivalently, the cofinite limit of the ratios vanishes:

$$\lim_{n\in N_0(T)} \frac{\Phi(\mu(T^{-m}(\{n\})))}{\Phi(\mu(\{n\}))} = 0$$

in the sense that for every $\varepsilon > 0$, all but finitely many $n \in N_0(T)$ satisfy the inequality. If there is no $m \in \mathbb{N}$ such that the above condition is satisfied, then we put $W\mathrm{ent}_\Phi(T) = \infty$.
For details about the above definitions we refer to [3, 4].

Lemma 3.1 *The following are equivalent:*
$|\mathrm{Per}(T)| < \infty \Leftrightarrow |W\mathrm{per}(T)| < \infty, \quad |N_U(T)| < \infty$.
Proof. Suppose $|\mathrm{Per}(T)| < \infty$. Then $W\mathrm{per}(T) \subseteq \mathrm{Per}(T)$ is finite. Also

$$N_U(T) = \bigcup_{k\in U\mathrm{per}(T)} N_k(T)$$

is a finite union of finite sets, hence finite.
Conversely, suppose $|W\mathrm{per}(T)| < \infty$ and $|N_U(T)| < \infty$.
If $U\mathrm{per}(T)$ were infinite, then since $N_k(T) \ne \varnothing$ for each $k \in U\mathrm{per}(T)$ and the sets $N_k(T)$ are pairwise disjoint, the union $N_U(T)$ would be infinite. Contradiction.
Hence $U\mathrm{per}(T)$ is finite, so $\mathrm{Per}(T) = W\mathrm{per}(T) \cup U\mathrm{per}(T)$ is finite. □

4 Norm Estimates for Characteristic Functions

Lemma 4.1 *Let A, B be disjoint with $\mu(A) = a$,*

$\mu(B) = b$, $a \le b$. Put $e_A = \chi_A/\Phi(a)$, $e_B = \chi_B/\Phi(b)$.

Then for $q < \infty$, $\| e_A - e_B \|^q_{p,q,b,\mu} = 1 + \frac{\Phi(a+b)^q - \Phi(a)^q}{\Phi(b)^q}$.

In particular, if $a = b$, then $\| e_A - e_B \|^q = \left(\frac{\Phi(2a)}{\Phi(a)}\right)^q$.

Proof. The decreasing rearrangement of $e_A - e_B$ is

$$(e_A - e_B)^*(t) = \begin{cases} \dfrac{1}{\Phi(a)}, & 0 < t < a, \\ \dfrac{1}{\Phi(b)}, & a < t < a+b, \\ 0, & t \ge a+b. \end{cases}$$

Substituting this into the Lorentz–Karamata norm gives the stated formula. □

Remark 4.2 Because Φ is doubling (Corollary 2.7), Lemma 4.1 implies that normalized differences of disjoint characteristic functions are bounded away from zero uniformly when a and b are comparable. Explicitly, if $a = b$, then

$$\| e_A - e_B \| \ge \left(\frac{\Phi(2a)}{\Phi(a)}\right) \ge c_0 > 0,$$

independently of a.

Lemma 4.3 (Compression estimate). *Let $A \subseteq \mathbb{N}$ be a set of atoms, and let $c \in (0,1]$.*

Let $f = \sum_{n\in A} a_n \chi_{\{n\}} \in LK^{p,q}_{b,\mu}$ be supported on A, and suppose that $\mu\left(T^{-m}(\{n\})\right) \le c\,\mu(\{n\}) \ (n \in A)$.

Then $\| C_T^m f \|_{p,q,b,\mu} \le C_\delta\, c^{1/p-\delta} \ \| f \|_{p,q,b,\mu}$,

where $\delta > 0$ is arbitrary and C_δ depends only on δ, p, q, b.
Proof. For $\lambda > 0$, set $A_\lambda = \{n \in A: |a_n| > \lambda\}$. Then

$$\mu(\{|C_T^m f| > \lambda\}) = \mu(T^{-m}(A_\lambda)) = \sum_{n\in A_\lambda} \mu\,(T^{-m}(\{n\})) \le c \sum_{n\in A_\lambda} \mu\,(\{n\}) = c\,\mu(\{|f| > \lambda\}).$$

Hence $(C_T^m f)^*(t) \le f^*(t/c)$ for all $t > 0$. Therefore

$$\| C_T^m f \|^q = \int_0^\infty (t^{1/p} b(t)(C_T^m f)^*(t))^q \frac{dt}{t} \le \int_0^\infty (t^{1/p} b(t) f^*(t/c))^q \frac{dt}{t}.$$

Substituting $s = t/c$ gives

$$= c^{q/p} \int_0^\infty \left(s^{1/p} b(cs) f^*(s)\right)^q \frac{ds}{s}.$$

By Potter's bound, $b(cs)^q \le C_\delta^q c^{-q\delta} b(s)^q$ for $c \le 1$ and any $\delta > 0$. Hence
$\| C_T^m f \|^q \le C_\delta^q \, c^{q(1/p-\delta)} \, \| f \|^q$.
Taking q-th roots gives the result. □

5 Disjointness of Preimage Trees

Lemma 5.1 *Let $n \in \mathbb{N}$ be aperiodic, i.e. n is not periodic. Then for $i \ne i'$, $T^{-i}(\{n\}) \cap T^{-i'}(\{n\}) = \varnothing$.*

Proof. Suppose $x \in T^{-i}(\{n\}) \cap T^{-i'}(\{n\})$ with $i < i'$. Then $T^i(x) = T^{i'}(x) = n$. Applying $T^{i'-i}$ to the first equality gives $T^{i'}(x) = T^{i'-i}(n)$. Hence $T^{i'-i}(n) = n$, so n is periodic with period $i' - i$, contradiction. □

Lemma 5.2 *Suppose T satisfies Assumption 2.2 and $N_0(T)$ is infinite. Let $t \ge 0$. Then for every infinite set $S \subseteq N_0(T)$ there exists a sequence $\{n_j\}_{j=1}^\infty \subset S$ such that the sets $T^{-i}(\{n_j\}), \quad i = 0,1,\dots,t, \quad j = 1,2,\dots$, are pairwise disjoint.*

Proof. We construct the sequence inductively. Since S is infinite, choose $n_1 \in S$ arbitrarily. Suppose $n_1, \dots, n_{k-1}$ have been chosen. Define the finite set

$$B_k = \bigcup_{j=1}^{k-1} \left(\bigcup_{i=0}^{t} T^i(\{n_j\}) \cup \bigcup_{i=1}^{t} T^{-i}(\{n_j\}) \right).$$

Each $T^i(\{n_j\})$ is a single point (or empty), and each $T^{-i}(\{n_j\})$ is finite by Assumption 2.2. Since there are finitely many j and finitely many i, the set B_k is finite. Since S is infinite, choose $n_k \in S \backslash B_k$.

We claim that the sets $T^{-i}(\{n_j\})$ for $i = 0, \dots, t$ and $j = 1,2,\dots$ are pairwise disjoint. Suppose $x \in T^{-i_1}(\{n_k\}) \cap T^{-i_2}(\{n_j\})$ for some $j < k$ and $i_1, i_2 \in \{0, \dots, t\}$. Then $T^{i_1}(x) = n_k$ and $T^{i_2}(x) = n_j$. If $i_1 \ge i_2$, then $T^{i_1-i_2}(n_j) = T^{i_1-i_2}(T^{i_2}(x)) = T^{i_1}(x) = n_k$. So $n_k \in T^{i_1-i_2}(\{n_j\})$ with $i_1 - i_2 \in \{0, \dots, t\}$, hence $n_k \in B_k$, contradiction.

If $i_1 < i_2$, then $T^{i_2-i_1}(n_k) = T^{i_2-i_1}(T^{i_1}(x)) = T^{i_2}(x) = n_j$. So $n_k \in T^{-(i_2-i_1)}(\{n_j\})$ with $i_2 - i_1 \in \{1, \dots, t\}$, hence $n_k \in B_k$, contradiction.

Therefore the sets are pairwise disjoint. □

6 Essential Characteristic Polynomial

Lemma 6.1 *Let n be periodic with cycle length s, so that $n, T(n), \dots, T^{s-1}(n)$ are distinct and $T^s(n) = n$. Let $p(z) = \sum_{j=0}^t p_j z^j$. For $r = 0, \dots, s-1$, put $\alpha_r = \sum_{j \equiv -r \pmod s} p_j$. Let $m_r = \mu(\{T^r(n)\})$, $r = 0, \dots, s-1$. Then $p(C_T)e_{\{n\}} = \frac{1}{\Phi(m_0)} \sum_{r=0}^{s-1} \alpha_r \, \chi_{\{T^r(n)\}}$.*

Moreover, $\| p(C_T)e_{\{n\}} \| \ge \frac{\max_{0 \le r < s} |\alpha_r|}{\Phi(m_0)} \min_{0 \le r < s} \Phi(m_r)$.
If Assumption 2.3 holds, then $\| p(C_T)e_{\{n\}} \| \ge c_K \max_{0 \le r < s} |\alpha_r|$.
Finally, the algebraic identity $\alpha_0 = \dots = \alpha_{s-1} = 0 \Leftrightarrow z^s - 1 \mid p(z)$ holds. More precisely, if $p(z) = (z^s - 1)h(z) + R(z)$ with $\deg R < s$, then $R(z) = \sum_{r=0}^{s-1} \alpha_r z^r$.

Proof. We have

$$p(C_T)e_{\{n\}} = \frac{1}{\Phi(m_0)} \sum_{j=0}^{t} p_j \, \chi_{T^{-j}(\{n\})}.$$

On the cycle, the coefficient at the point $T^r(n)$ is precisely α_r. Since the points $T^r(n)$ are distinct, the functions $\chi_{\{T^r(n)\}}$ are disjointly supported. Hence

$$|p(C_T)e_{\{n\}}| \ge \frac{\max_r |\alpha_r|}{\Phi(m_0)} \chi_{\{T^{r_0}(n)\}},$$

where r_0 is an index at which the maximum is attained. Taking norms and using monotonicity gives the first bound. For the second bound, by Assumption 2.3 and Corollary 2.7, iterating the doubling inequality a finite number of times yields $\Phi(m_r) \ge c_K \Phi(m_0)$ uniformly in r, where $c_K > 0$ depends only on K and the doubling constant.

For the algebraic identity, divide p by $z^s - 1$ with remainder:
$p(z) = (z^s - 1)h(z) + R(z), \qquad \deg R < s$.
Write $R(z) = \sum_{r=0}^{s-1} \alpha_r z^r$. Since $T^r(n) = T^{r+s}(n)$ for all r, the coefficient of z^r in R equals the sum of the coefficients p_j with $j \equiv r \pmod s$, which is α_r. Hence $R \equiv 0$ iff all $\alpha_r = 0$, i.e. iff $z^s - 1 \mid p(z)$. □

Lemma 6.2 *Suppose $s \in W\mathrm{per}(T)$. Then there exists a sequence $\{n_j\}_{j=1}^\infty$ of periodic points with period s such that the cycles $\{n_j, T(n_j), \dots, T^{s-1}(n_j)\}$ are pairwise disjoint. Moreover, if $p(z)$ is a polynomial such that $z^s - 1 \nmid p(z)$, then the sequence $f_j = \frac{\chi_{\{n_j\}}}{\Phi(\mu(\{n_j\}))}$ satisfies $\| f_j \| = 1$ and there is a constant $c > 0$, independent of j, such that $\| p(C_T) f_j \| \ge c$ for all j. Hence $p(C_T)$ cannot be compact.*

Proof. Since $s \in W\mathrm{per}(T)$, the set $N_s(T)$ is infinite. We extract a sequence of pairwise disjoint s-cycles greedily: choose $n_1 \in N_s(T)$; since $N_s(T)$ is infinite, choose $n_2 \in N_s(T)$ outside the finite cycle of n_1; continue inductively. For the periodic point n_j, the entire inverse orbit is contained in the cycle:
$T^{-r}(\{n_j\}) \subseteq \{n_j, T(n_j), \dots, T^{s-1}(n_j)\} \qquad (r \ge 0)$.

Hence the supports of $p(C_T)f_j$ are contained in the disjoint cycles. For each j, Lemma 6.1 and Assumption 2.3 give
$\| p(C_T) f_j \| \ge c_K \max_{0 \le r < s} |\alpha_r|$,

where $\alpha_r = \sum_{\ell \equiv -r \pmod s} p_\ell$. If $z^s - 1 \nmid p(z)$, then not all α_r are zero, so $\max_r |\alpha_r| > 0$, and the lower bound is uniform in j with constant $c = c_K \max_r |\alpha_r| > 0$. Since the cycles are pairwise disjoint, the functions $p(C_T)f_j$ and $p(C_T)f_k$ are supported on disjoint sets for $j \ne k$. Applying Lemma 4.1 and Remark 4.2 with $a = b = \mu(\{n_j\})$ (after normalization), there is a constant $\delta > 0$, independent of j, k, such that
$\| p(C_T) f_j - p(C_T) f_k \| \ge \delta > 0$.

Thus the sequence $\{p(C_T)f_j\}$ has no convergent subsequence, so $p(C_T)$ is not compact. □

Lemma 6.3 *Suppose $m < W\mathrm{ent}_\Phi(T)$. Then there exist infinitely many $n \in N_0(T)$ such that $\frac{\Phi(\mu(T^{-m}(\{n\})))}{\Phi(\mu(\{n\}))} \not\to 0$. If $p(z) = p_m z^m + \dots + p_t z^t$ with $p_m \ne 0$, then along a suitable sequence, $\| p(C_T)e_{\{n\}} \| \ge c \, |p_m| \, \frac{\Phi(\mu(T^{-m}(\{n\})))}{\Phi(\mu(\{n\}))}$. Hence $p(C_T)$ cannot be compact. Therefore, if $p(C_T)$ is compact, then $p_m = 0$ for every $m < W\mathrm{ent}_\Phi(T)$; equivalently, $z^{W\mathrm{ent}_\Phi(T)} \mid p(z)$.*

Proof. Since $m < W\mathrm{ent}_\Phi(T)$, there is an $\varepsilon > 0$ and an infinite set $S \subset N_0(T)$ such that

$$\frac{\Phi(\mu(T^{-m}(\{n\})))}{\Phi(\mu(\{n\}))} \ge \varepsilon \qquad (n \in S).$$

Otherwise the cofinite limit defining $W\mathrm{ent}_\Phi(T)$ would vanish for this m, contradicting $m < W\mathrm{ent}_\Phi(T)$. Apply Lemma 5.2 with t equal to the degree of p and with the infinite set S, obtaining a sequence $\{n_j\} \subset S$ such that the sets
$T^{-i}(\{n_j\}), \qquad i = 0,1,\dots,t, \qquad j = 1,2,\dots,$
are pairwise disjoint.
By Lemma 5.1, for each fixed j the sets $T^{-i}(\{n_j\})$ for $i = 0, \dots, t$ are pairwise disjoint. Lemma 5.2 ensures disjointness across different j.
For each j,

$$p(C_T)e_{\{n_j\}} = \frac{1}{\Phi(\mu(\{n_j\}))} \sum_{i=0}^{t} p_i \, \chi_{T^{-i}(\{n_j\})}.$$

The term with $i = m$ has coefficient $p_m / \Phi(\mu(\{n_j\}))$ and is supported on $T^{-m}(\{n_j\})$, which is disjoint from all other terms. Hence

$$\| p(C_T)e_{\{n_j\}} \| \geq |p_m| \frac{\| \chi_{T^{-m}(\{n_j\})} \|}{\Phi\left(\mu(\{n_j\})\right)}$$
$$= |p_m| \frac{\Phi(\mu(T^{-m}(\{n_j\})))}{\Phi(\mu(\{n_j\}))} \geq |p_m|\varepsilon.$$

This is a uniform lower bound. The sequence $e_{\{n_j\}}$ is bounded (norm 1), and the images $p(C_T)e_{\{n_j\}}$ have disjoint supports, so by the lattice property,

$$\| p(C_T)e_{\{n_j\}} - p(C_T)e_{\{n_k\}} \| \geq |p_m|\varepsilon \qquad (j \neq k).$$

Thus no subsequence converges, and $p(C_T)$ is not compact. □

7 The Key Compactness Lemma

Lemma 7.1 *Let $T: \mathbb{N} \to \mathbb{N}$ be a self-map satisfying Assumptions 2.2 and 2.3.Assume that $b \in SV_0 \cap SV_\infty$ and that $C_T \in \mathcal{B}(LK^{p,q}_{b,\mu})$.*
If $m = W\mathrm{ent}_\Phi(T) < \infty$ and $|\mathrm{Per}(T)| < \infty$, then $q(C_T) = C_T^m \prod_{k \in W\mathrm{per}(T)} (C_T^k - I)$ is compact.

Proof. Decompose $\mathbb{N} = N_{per} \sqcup N_{ap}$ with

$$N_{per} = \{n: \mathrm{ent}_T(n) = 0\}, \qquad N_{ap} = \{n: \mathrm{ent}_T(n) \geq 1\}.$$

Both sets are forward T-invariant: if $n \in N_{per}$, then $\mathrm{ent}_T(n) = 0$ and hence $\mathrm{ent}_T(T(n)) = 0$, so $T(n) \in N_{per}$; similarly for N_{ap}. Since $LK^{p,q}_{b,\mu}$ is a Banach lattice with order-continuous norm, the multiplication projections

$$P_{per}\chi_{\{n\}} = \chi_{\{n\}} \text{for} n \in N_{per}, \qquad P_{ap}\chi_{\{n\}} = \chi_{\{n\}} \text{for} n \in N_{ap},$$

are bounded, and
$LK^{p,q}_{b,\mu} = P_{per}X \oplus P_{ap}X.$

Periodic part.
Decompose $P_{per}X$ by cycle length. For $k \in \mathrm{Per}(T)$, let
$N_k = \{n \in N_{per}: \mathrm{cyc}_T(n) = k\}, \qquad X_k = \overline{\mathrm{span}}\{\chi_{\{n\}}: n \in N_k\}.$

Then
$P_{per}X = \oplus_{k \in \mathrm{Per}(T)} X_k.$
Since $|\mathrm{Per}(T)| < \infty$, this is a finite direct sum of invariant subspaces.
On X_k with $k \in W\mathrm{per}(T)$, we have $C_T^k = I$, so $C_T^k - I = 0$. Hence

$$\prod_{k' \in W\mathrm{per}(T)} (C_T^{k'} - I)|_{X_k} = 0.$$

On X_k with $k \in U\mathrm{per}(T)$, the set N_k is finite, so X_k is finite-dimensional, and every operator on it is compact.
Therefore $q(C_T)|_{P_{per}X}$ is compact.

Aperiodic part. We show that $C_T^m|_{P_{ap}X}$ is compact. Since

$$q(C_T)|_{P_{ap}X} = C_T^m \left(\prod_{k \in W\mathrm{per}(T)} (C_T^k - I) \right)|_{P_{ap}X},$$

and the product is bounded while compact operators form a two-sided ideal, it suffices to prove that $C_T^m|_{P_{ap}X}$ is compact.
For $\varepsilon > 0$, define

$$F_\varepsilon = \left\{ n \in N_0(T): \frac{\Phi(\mu(T^{-m}(\{n\})))}{\Phi(\mu(\{n\}))} \geq \varepsilon \right\}.$$

By the definition of $W\mathrm{ent}_\Phi(T) = m$, the set F_ε is finite for every $\varepsilon > 0$.
Let P_ε be the projection onto $\mathrm{span}\{\chi_{\{n\}}: n \in F_\varepsilon\}$, and set $Q_\varepsilon = I - P_\varepsilon$ on $P_{ap}X$. The projection P_ε has finite rank.
We estimate $\| C_T^m Q_\varepsilon f \|$ for $f \in P_{ap}X$.
Write $f = \sum_{n \in A} a_n \chi_{\{n\}}$ with $A \subseteq N_0(T) \setminus F_\varepsilon$ (by definition of $P_{ap}X$, f is supported on $N_0(T)$). For $n \in A$, we have $\Phi(\mu(T^{-m}(\{n\}))) < \varepsilon\, \Phi(\mu(\{n\}))$.

By Lemma 2.8(ii), there exists $\eta(\varepsilon) > 0$ with $\eta(\varepsilon) \to 0$ as $\varepsilon \to 0$ such that

$$\mu(T^{-m}(\{n\})) < \eta(\varepsilon)\, \mu(\{n\}) \qquad (n \in A).$$

Set $c = \eta(\varepsilon) \in (0,1]$ for ε sufficiently small. By Lemma 4.3,

$$\| C_T^m f \| \leq C_\delta\, c^{1/p-\delta} \| f \| = C_\delta\, \eta(\varepsilon)^{1/p-\delta} \| f \|.$$

Hence
$$\| C_T^m Q_\varepsilon \| \leq C_\delta\, \eta(\varepsilon)^{1/p-\delta} \to 0 \qquad (\varepsilon \to 0).$$

Therefore
$$C_T^m|_{P_{ap}X} = C_T^m P_\varepsilon|_{P_{ap}X} + C_T^m Q_\varepsilon|_{P_{ap}X},$$

where the first term has finite rank and the second has norm tending to zero as $\varepsilon \to 0$. Hence $C_T^m|_{P_{ap}X}$ is a norm limit of finite-rank operators and is therefore compact.
Consequently, $q(C_T)$ is compact on both $P_{per}X$ and $P_{ap}X$, hence on all of $LK^{p,q}_{b,\mu}$. □

Main Classification Theorem

Theorem 25. *Let $T: \mathbb{N} \to \mathbb{N}$ be a self-map satisfying Assumptions 2.2 and 2.3. Assume that $b \in SV_0 \cap SV_\infty$ and that $C_T \in \mathcal{B}(LK^{p,q}_{b,\mu})$.*
Then the following are equivalent:

$$\mathrm{dimalg}_\pi(C_T) < \infty$$
$$\Updownarrow$$
$$W\mathrm{ent}_\Phi(T) < \infty, \qquad |\mathrm{Per}(T)| < \infty$$
$$\Updownarrow$$
$$W\mathrm{ent}_\Phi(T) < \infty, \qquad |W\mathrm{per}(T)| < \infty, \qquad |N_U(T)| < \infty.$$

Moreover, if $\mathrm{alg}_\pi(C_T)$ is finite-dimensional, then $\mathrm{alg}_\pi(C_T) \cong \mathbb{C}[z]/(q(z))$, where $q(z) = z^m \mathrm{lcm}_{k \in W\mathrm{per}(T)}(z^k - 1)$, with $m = W\mathrm{ent}_\Phi(T)$.

Proof. The equivalence of the second and third conditions is Lemma 3.1.

Necessity. Suppose $\mathrm{dimalg}_\pi(C_T) < \infty$.
Then there is a nonzero polynomial p such that $p(C_T)$ is compact. By Lemma 6.2, p must be divisible by $z^k - 1$ for every $k \in W\mathrm{per}(T)$. By Lemma 6.3, p must be divisible by z^m for every $m < W\mathrm{ent}_\Phi(T)$. Hence

$$z^{W\mathrm{ent}_\Phi(T)} \mathrm{lcm}_{k \in W\mathrm{per}(T)}(z^k - 1) \mid p(z).$$
If either $W\mathrm{ent}_\Phi(T) = \infty$ or $|W\mathrm{per}(T)| = \infty$, then no nonzero polynomial can satisfy these divisibility conditions, contradiction. Hence $W\mathrm{ent}_\Phi(T) < \infty$ and $|W\mathrm{per}(T)| < \infty$. By Lemma 3.1, this is equivalent to the second condition.

Sufficiency. Suppose $W\mathrm{ent}_\Phi(T) < \infty$ and $|\mathrm{Per}(T)| < \infty$. By Lemma 7.1, the polynomial
$q(z) = z^{W\mathrm{ent}_\Phi(T)} \mathrm{lcm}_{k \in W\mathrm{per}(T)}(z^k - 1)$
satisfies $q(C_T) \in \mathcal{K}(LK^{p,q}_{b,\mu})$. Hence $q(\pi(C_T)) = 0$, so $\pi(C_T)$ is algebraic in the Calkin algebra.
Consider the algebra homomorphism
$\mathbb{C}[z] \to \mathrm{alg}_\pi(C_T), \qquad p \mapsto p(\pi(C_T)).$

Since $q(\pi(C_T)) = 0$, the kernel is a nonzero principal ideal of $\mathbb{C}[z]$, say (q_0). By the necessity part, every annihilating polynomial is divisible by
$z^{W\mathrm{ent}_\Phi(T)} \mathrm{lcm}_{k \in W\mathrm{per}(T)}(z^k - 1),$

so this polynomial has the least degree among the annihilators and is thus the monic generator of the kernel.
Hence $q_0 = q$, and by the first isomorphism theorem, $\mathrm{alg}_\pi(C_T) \cong \mathbb{C}[z]/(q)$. □

Corollary 8.2 *Under the hypotheses of Theorem 8.1,* $C_T \in \mathcal{K}(LK^{p,q}_{b,\mu})$ *if and only if*

$Went_\Phi(T) = 0,\ \ Wper(T) = \varnothing,\ \ |Per(T)| < \infty.$

Equivalently, C_T is compact if and only if the minimal annihilating polynomial of $\pi(C_T)$ in the Calkin algebra is 1.

Proof. C_T is compact iff $\pi(C_T) = 0$ iff the minimal annihilating polynomial is $q(z) = 1$. By Theorem 8.1, $q(z) = 1$ iff $Went_\Phi(T) = 0$, $Wper(T) = \varnothing$, and $|Per(T)| < \infty$. □

9 Invariance Under Slowly Varying Perturbation

Proposition 9.1 *Let $b \in SV_0 \cap SV_\infty$. For positive sequences*

x_n, y_n with $x_n, y_n \to 0$ and $x_n \le y_n$,
$\lim_{n\to\infty} \frac{\Phi(x_n)}{\Phi(y_n)} = 0 \Leftrightarrow \lim_{n\to\infty} \frac{x_n}{y_n} = 0$. *The same holds if $x_n, y_n \to \infty$.*

Proof. By Proposition 2.6,

$$\Phi(s) \sim \left(\frac{p}{q}\right)^{1/q} s^{1/p} b(s) \qquad (s \to 0 ors \to \infty).$$

Hence

$$\frac{\Phi(x_n)}{\Phi(y_n)} \asymp \left(\frac{x_n}{y_n}\right)^{1/p} \frac{b(x_n)}{b(y_n)}.$$

By the Potter bounds for slowly varying functions [2, Theorem 1.5.6], for every $\varepsilon > 0$ there is a constant $C_\varepsilon > 0$ such that

$$\frac{b(x)}{b(y)} \le C_\varepsilon \left(\frac{y}{x}\right)^{\varepsilon} \qquad (x \le y).$$

Choosing $\varepsilon < 1/p$ yields

$$\frac{\Phi(x_n)}{\Phi(y_n)} \le C \left(\frac{x_n}{y_n}\right)^{1/p-\varepsilon} \to 0$$

whenever $x_n/y_n \to 0$. The converse follows from the reverse Potter bound

$$\frac{\Phi(x_n)}{\Phi(y_n)} \ge c \left(\frac{x_n}{y_n}\right)^{1/p+\varepsilon}$$

for some $\varepsilon > 0$. □

Proposition 9.2. *Let $b \in SV_0 \cap SV_\infty$. For positive sequences x_n, y_n with $c \le \frac{x_n}{y_n} \le C$ for all n, there are constants $c', C' > 0$ such that $c' \le \frac{\Phi(x_n)}{\Phi(y_n)} \le C'$ for all n.*

Proof. By the doubling condition (Corollary 2.7), iterating the inequality $\Phi(2s) \le C_0\Phi(s)$ a finite number of times gives $\Phi(Cs) \le C'\Phi(s)$ for any fixed $C > 0$. Similarly, iterating the lower bound gives $\Phi(Cs) \ge c'\Phi(s)$. Applying these with C and $1/c$ to $x_n = (x_n/y_n)y_n$ and $y_n = (y_n/x_n)x_n$ yields the result. □

Theorem 9.3 *Let $T: \mathbb{N} \to \mathbb{N}$ be a self-map satisfying Assumptions 2.2 and 2.3, and assume that for every $m \ge 1$ and every sequence $n_j \in N_0(T)$,*

$$\frac{\mu(T^{-m}(\{n_j\}))}{\mu(\{n_j\})} \to 0 \Leftrightarrow \frac{\Phi(\mu(T^{-m}(\{n_j\})))}{\Phi(\mu(\{n_j\}))} \to 0.$$

Then $Went_\Phi(T) = Went(T)$, where $Went(T)$ is the essential enter length in the classical Lorentz space $L^{p,q}_\mu$. Consequently, the classification of essentially algebraic composition operators on $LK^{p,q}_{b,\mu}$ is identical to that on $L^{p,q}_\mu$.

Proof. The hypothesis states precisely that the cofinite limit defining $Went_\Phi(T)$ vanishes if and only if the cofinite limit defining $Went(T)$ vanishes, for every m. Hence the minimal such m is the same in both cases. The remaining combinatorial data $Per(T)$, $Wper(T)$, and $N_U(T)$ are independent of the space. Therefore the classification theorem and the compactness corollary are the same. □

Corollary 9.4 *The hypothesis of Theorem 9.3 is satisfied if, for every $m \ge 1$, the ratios $r_n^{(m)} = \frac{\mu(T^{-m}(\{n\}))}{\mu(\{n\})}$ either tend to 0, or satisfy $\liminf_{n\in N_0(T)} r_n^{(m)} > 0$, as $n \to \infty$ along $N_0(T)$.*

Proof. This follows from Propositions 9.1 and 9.2. □

Remark 9.5 The slowly varying function b does not change the zero/nonzero nature of the asymptotic ratios that determine compactness and essential algebraicity. The norm itself changes when b changes, but the algebraic classification of composition operators remains the same as in the classical Lorentz case, under the stated regular-variation hypotheses.

10 Examples

Example 10.1 (Pure periodic case).
Let T consist of infinitely many cycles of length k. Then

$$Wper(T) = \{k\}, \qquad Per(T) = \{k\}, \qquad N_U(T) = \varnothing, \qquad N_0(T) = \varnothing.$$

Hence $Went_\Phi(T) = 0$, and $q(z) = z^k - 1$.

Thus
$alg_\pi(C_T) \cong \mathbb{C}[z]/(z^k - 1)$.

Example 10.2 (Infinite essential enter length). Let

$$T(n) = \begin{cases} n-1, & n \ge 2, \\ 1, & n = 1. \end{cases}$$

Then T is finite-to-one: $T^{-1}(\{1\}) = \{1,2\}$, and $T^{-1}(\{n\}) = \{n+1\}$ for $n \ge 2$. With counting measure,
$T^{-m}(\{n\}) = \{n+m\} \qquad (n \ge 2, m \ge 1)$,

so

$$\frac{\Phi(\mu(T^{-m}(\{n\})))}{\Phi(\mu(\{n\}))} = \frac{\Phi(1)}{\Phi(1)} = 1.$$

Hence the limit defining $Went_\Phi(T)$ does not vanish for any m, so
$Went_\Phi(T) = \infty$.
This example shows that finite-to-one maps can have infinite essential enter length.

Example 10.3 (Finite essential enter length).
Let $T(2^k) = 2^{k-1}$ for $k \ge 1$, and $T(n) = n$ for n not a power of 2. Then $T(2) = 1, T(4) = 2, T(8) = 4$, and so on. The point 1 is fixed,
$2 \to 1, 4 \to 2 \to 1, 8 \to 4 \to 2 \to 1$, etc.
Define the weight

$$\mu(\{2^k\}) = 2^{-2^k} \quad (k \ge 1), \qquad \mu(\{n\}) = 1 \quad (n \notin \{2^k : k \ge 1\}).$$

Then T is finite-to-one. For $n = 2^k$,

$$T^{-m}(\{2^k\}) = \{2^{k+m}\},$$

so

$$\frac{\mu(T^{-m}(\{2^k\}))}{\mu(\{2^k\})} = \frac{2^{-2^{k+m}}}{2^{-2^k}} = 2^{-(2^{k+m}-2^k)} = 2^{-2^k(2^m-1)} \to 0 \ \ (k \to \infty).$$

For $m = 0$, the ratio is 1. Hence the cofinite limit defining $Went_\Phi(T)$ vanishes precisely for $m \ge 1$, so

$Went_\Phi(T) = 1$.

To see that C_T is bounded, we check the criterion of Theorem 2.9. For a singleton $\{n\}$ with n not a power of 2, $T^{-1}(\{n\}) = \{n\}$ and $\mu(T^{-1}(\{n\})) = \mu(\{n\})$, so the ratio is 1. For $\{2^k\}$,

$$\frac{\mu(T^{-1}(\{2^k\}))}{\mu(\{2^k\})} = \frac{\mu(\{2^{k+1}\})}{\mu(\{2^k\})} = \frac{2^{-2^{k+1}}}{2^{-2^k}} = 2^{-2^k} \le 1.$$

For $\{1\}$, $T^{-1}(\{1\}) = \{1,2\}$ so

$$\frac{\mu(T^{-1}(\{1\}))}{\mu(\{1\})} = \frac{1+2^{-2}}{1} = 1 + 1/4 = 5/4.$$

For a general set $A \subseteq \mathbb{N}$, write $A = A_{np} \cup A_p$ where A_{np} contains non-powers of 2 and A_p contains powers of 2. Then

$$\mu(T^{-1}(A)) = \mu(A_{np}) + \sum_{2^k \in A_p} \mu\left(\{2^{k+1}\}\right) + \begin{cases} \mu(\{2\}), & 1 \in A, \\ 0, & 1 \notin A. \end{cases}$$

Since $\mu(\{2^{k+1}\}) \le \mu(\{2^k\})$ for all $k \ge 1$ and $\mu(\{2\}) = 2^{-2} < 1 = \mu(\{1\})$, we obtain

$$\mu(T^{-1}(A)) \le \mu(A_{np}) + \sum_{2^k \in A_p} \mu\left(\{2^k\}\right) + \mu(\{1\}) \cdot \mathbf{1}_{1 \in A} = \mu(A).$$

Hence the ratio $\mu(T^{-1}(A))/\mu(A) \le 1$ for all A, and by monotonicity of Φ, the supremum in Theorem 2.9 is finite. Therefore C_T is bounded on $LK^{p,q}_{b,\mu}$.

The essential characteristic polynomial is
$q(z) = z(z-1)$,
since $W\mathrm{per}(T) = \{1\}$ (all fixed points have cycle length 1, and there are infinitely many of them).

Example 10.4 (Slowly varying perturbation). Let

$$b(t) = (1 + |\log t|)^{\alpha}, \qquad \alpha \in \mathbb{R}.$$

Then $b \in SV_0 \cap SV_\infty$, and

$$\Phi(s) \asymp s^{1/p}(1 + |\log s|)^{\alpha}.$$

For the explicit sequences

$$x_n = e^{-n^2}, \qquad y_n = e^{-n},$$

we have $x_n/y_n \to 0$ and

$$\frac{\Phi(x_n)}{\Phi(y_n)} \asymp \left(\frac{x_n}{y_n}\right)^{1/p} \left(\frac{1 + |\log x_n|}{1 + |\log y_n|}\right)^{\alpha} \to 0.$$

So the slowly varying factor does not change the zero/nonzero nature of the limit.

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