Discover Engineering
An International Journal
OPEN ACCESS
ISSN 3117-566X
Original Research Article
Check for updates

A diffeomorphic approach to interpolating prime series and solving diophantine equations

Ramaraju Yadav E*, and Chandra Bhan Singh

Department of Mathematics, Sunrise University, Alwar, Rajasthan 301001, India

Corresponding author: **Ramaraju Yadav E** | E-mail: **yadavramaraju@gmail.com**

Citation: Ramaraju Yadav E, and Chandra Bhan Singh (2025). A diffeomorphic approach to interpolating prime series and solving diophantine equations. *Discover Engineering: An International Journal.* **DOI: https://doi.org/10.51470/DE.2025.6.1.27**

Received 19 January 2025 | Revised 20 February 2025 | Accepted 21 March 2025 | Available Online 23 April 2025

Abstract

A diffeomorphic pair $(p(x), p^{-1}(x))$ is constructed in this work by interpolating the prime series with a suitable quadric spline. The prime counting function and the prime series are smooth analogues of these continuously differentiable functions, which capture the key behaviour of prime numbers. Estimates for Diophantine equations over primes are obtained by using the function $p^{-1}(x)$ The research also investigates the possibility of using this function to the analysis of the prime number theorem's error term in the von Koch estimate; this theorem is closely related to the Riemann Hypothesis.

Keywords: *Diophantine Equations; Diffeomorphic Approach; prime series*

Introduction

Let N represent the set of natural numbers. Let p_n or $p(n)$ denote the n-th prime number. Let $\hat{p}_n$represent the number of composites in the interval $p_{n+1} - p_n - 1$. Let $\mathbb{P}$ be the set of prime numbers. Let π(x) denote the prime counting function. Let Li(x) represent the logarithmic integral, which is defined as follows:

$$\mathbb{N} = \{1,2,3,\dots\},\ \mathbb{P} = \{p(n): n \in \mathbb{N}\},\ \pi(x) = \sum_{p \le x, p \in \mathbb{P}} 1,\ \mathrm{Li}(x) = \int_2^x \frac{ds}{\ln(s)}$$

When it comes to solving nonlinear issues, these diffeomorphisms enable us to employ iterative techniques and Fourier analysis in a natural approach. This is especially useful when accounting for the unique characteristics of the primes' non-asymptotic behaviour.

Quadric spline-based prime number diffeomorphisms

We may create these functions now.

$$a_n^-(x) = -2\hat{p}_{n-1}(x-n)^2 + (x-n) + p_n,\ x > 0,\ n = 2,3,\dots,$$
$$a_n^+(x) = 2\hat{p}_n\left(x - n - \frac{1}{2}\right)^2 + (2\hat{p}_n + 1)\left(x - n - \frac{1}{2}\right) + \frac{p_n + p_{n+1}}{2}.$$

Their derivatives are

$$\frac{da_n^-(x)}{dx} = 4\hat{p}_{n-1}(n-x) + 1,\ \frac{da_n^+(x)}{dx} = 4\hat{p}_n(x-n) + 1.$$

For any $n = 2,3,\dots$ the functions are sewed together **(Milgram, 2020).**

$$a_n^-(n) = a_n^+(n) = p_n \dots\dots (1.1)$$
$$\left.\frac{da_n^-(x)}{dx}\right|_{x=n} = \left.\frac{da_n^+(x)}{dx}\right|_{x=n} = 1 \dots\dots (1.2)$$
$$a_{n+1}^-\left(n + \frac{1}{2}\right) = a_n^+\left(n + \frac{1}{2}\right) = \frac{1}{2}(p_n + p_{n+1}) \dots\dots (1.3)$$
$$\left.\frac{da_{n+1}^-(x)}{dx}\right|_{x=n+\frac{1}{2}} = \left.\frac{da_n^+(x)}{dx}\right|_{x=n+\frac{1}{2}} = 2\hat{p}_n + 1. \dots\dots (1.4)$$

A continually differentiable quadric spline is defined by equations (1.1), (1.2), (1.3), and (1.4).

© ***Authors:*** *Published in* ***Discover Engineering: An International Journal*** *under the CC BY-NC-ND 4.0 license (http://creativecommons.org/licenses/by-nc-nd/4.0/). No commercial use or modifications permitted.*

$$p(x)=\begin{cases} x+1, & 0<x\le\dfrac{3}{2} \\ a_n^-(x), & n-\dfrac{1}{2}\le x\le n, \quad n=2,3,\dots\dots\dots(1.5) \\ a_n^+(x), & n\le x\le n+\dfrac{1}{2}, \quad n=2,3,\dots \end{cases}$$

with first derivative

$$\frac{dp(x)}{dx}=\begin{cases} 1, & 0<x\le\dfrac{3}{2}, \\ 4\hat{p}_{n-1}(n-x)+1, & n-\dfrac{1}{2}\le x\le n, \quad n=2,3,\dots,\dots\dots(1.6) \\ 4\hat{p}_n(x-n)+1, & n\le x\le n+\dfrac{1}{2}, \quad n=2,3,\dots. \end{cases}$$

To get the following inverse functions and their derivatives, we may reverse both the functions $a_n^-(x)$ in the range $n-\frac{1}{2}\le x\le n$ and $a_n^+(x)$ in the interval $n\le x\le n+\frac{1}{2}$.

$$\begin{aligned} b_n^-(x) &= n+\frac{1-(8\hat{p}_{n-1}(p_n-x)+1)^{\frac{1}{2}}}{4\hat{p}_{n-1}} \\ b_n^+(x) &= n+\frac{(8\hat{p}_n(x-p_n)+1)^{\frac{1}{2}}-1}{4\hat{p}_n} \\ \frac{db_n^-(x)}{dx} &= (8\hat{p}_{n-1}(p_n-x)+1)^{-\frac{1}{2}} \\ \frac{db_n^+(x)}{dx} &= (8\hat{p}_n(x-p_n)+1)^{-\frac{1}{2}}. \end{aligned}$$

Equations (1.3), (1.4), (1.5), and (1.6) are analogous to the sewing together of the functions $b_n^-(x), b_n^+(x)$ and their derivatives:

$$\begin{aligned} b_n^-(p_n) &= b_n^+(p_n)=n,\dots\dots(1.7) \\ \left.\frac{db_n^-(x)}{dx}\right|_{x=p_n} &= \left.\frac{db_n^+(x)}{dx}\right|_{x=p_n}=1,\dots\dots(1.8) \\ b_{n+1}^-\left(\frac{p_n+p_{n+1}}{2}\right) &= b_n^+\left(\frac{p_n+p_{n+1}}{2}\right)=n+\frac{1}{2},\dots\dots(1.9) \\ \left.\frac{db_{n+1}^-(x)}{dx}\right|_{x=\frac{p_{n+p}p_{n+1}}{2}} &= \left.\frac{db_n^+(x)}{dx}\right|_{x=\frac{p_{n+1}}{2}}=\frac{1}{2\hat{p}_{n+1}+1}.\dots\dots(1.10) \end{aligned}$$

Finally, the constantly differentiable inverse spline with first derivative is defined by Eqs. (1.7), (1.8), (1.9), and (1.10).

$$\frac{dp^{-1}(x)}{dx}=\begin{cases} 1, & 1<x\le\dfrac{5}{2}, \\ (8\hat{p}_{n-1}(p_n-x)+1)^{-\frac{1}{2}}, & \dfrac{p_{n-1}+p_n}{2}\le x\le p_n, \quad n=2,3,\dots,\dots\dots(1.12) \\ (8\hat{p}_n(x-p_n)+1)^{-\frac{1}{2}}, & p_n\le x\le\dfrac{p_n+p_{n+1}}{2}, \quad n=2,3,\dots. \end{cases}$$

Lemma

Both $p(x)$ and $p^{-1}(x)$ have derivatives that meet the following inequality conditions.

$$1\le\frac{dp(x)}{dx}<\infty,\ x>0,$$
$$0<\frac{dp^{-1}(x)}{dx}\le 1,\ x>1.$$

Theorem

a) These two sets of prime integers are diffeomorphic: $(p(x),p^{-1}(x))$
b) Invariants follow the unique features of prime and counting curves.
c)

$$\begin{aligned} 1 &= \left.\frac{dp(x)}{dx}\right|_{x=n}=\left.\frac{dp^{-1}(x)}{dx}\right|_{x=p_n}, \ n=2,3,\dots, \\ -1 &= \operatorname{sign}\left(\left.\frac{d^2p(x)}{dx^2}\right|_{x=n-0}\right)\operatorname{sign}\left(\left.\frac{d^2p(x)}{dx^2}\right|_{x=n+0}\right), \ n=3,4,\dots, \\ -1 &= \operatorname{sign}\left(\left.\frac{d^2p^{-1}(x)}{dx^2}\right|_{x=p_n-0}\right)\operatorname{sign}\left(\left.\frac{d^2p^{-1}(x)}{dx^2}\right|_{x=p_n+0}\right), \ n=3,4,\dots \end{aligned}$$

Proof

a) A set of presences '$p^{-1}(x) \in C^{(1)}(1,\infty)$ and $p(x) \in C^{(1)}(0,\infty)$' are defined by definitions (1.5), (1.6), and (1.12). From equations (1.1) and (1.3), we may deduce that the interpolation conditions are valid. Since both $p(x)$ and $p^{-1}(x)$ are continuous and monotonically growing functions, and since the criteria can be verified explicitly, it follows that they are mutually invertible. The existence of a relationship between the functions '$\pi(x)$ and $p^{-1}(x)$' is demonstrated equations (1.7) [1].

b) After solving equations (1.2) and (1.8), we have equations (21). Second derivative discontinuity

$$\frac{d^2p(x)}{dx^2} = \begin{cases} 0, & 0 < x \le \frac{3}{2}, \\ -4\hat{p}_{n-1}, & n - \frac{1}{2} \le x \le n, \quad n = 2,3,\dots,\dots\dots (1.13) \\ 4\hat{p}_n, & n \le x \le n + \frac{1}{2}, \quad n = 2,3,\dots, \end{cases}$$

And

$$\frac{d^2p^{-1}(x)}{dx^2} = \begin{cases} 0, & 0 < x \le \frac{5}{2} \\ 4\hat{p}_{n-1}(8\hat{p}_{n-1}(p_n - x) + 1)^{-3/2}, & \frac{p_{n-1}+p_n}{2} \le x \le p_n \quad ,n = 2,3,\dots\dots\dots (1.14) \\ -4\hat{p}_n(8\hat{p}_n(x - p_n) + 1)^{-3/2}, & p_n \le x \le \frac{p_n + p_{n+1}}{2} \quad ,n = 2,3,\dots \end{cases}$$

is equivalent to Equations (1.13 and 1.14). The proof is now complete.

- **Remark**

The invertibility of p(x) and the fact that its coefficients are integers are both derived from Eq. (1.9) in Definition 1.

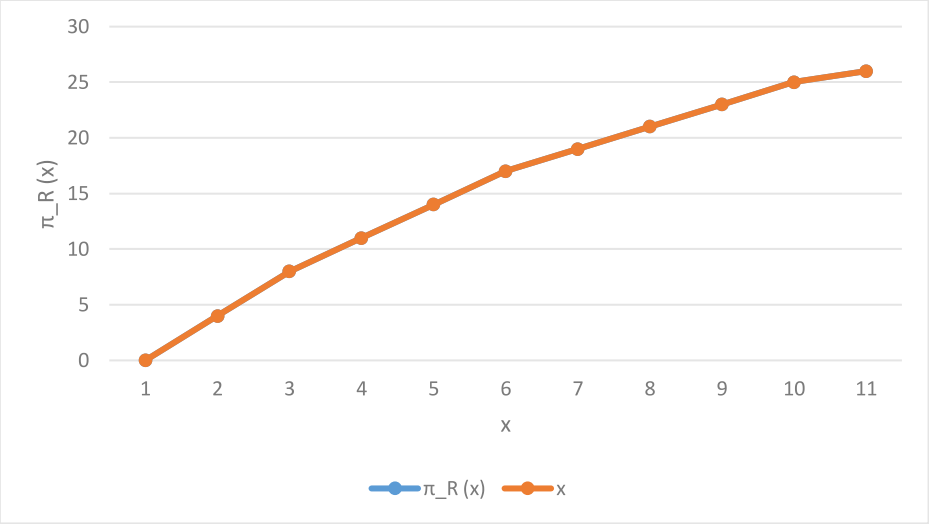


Figure 1.1 displays the Riemann-von Mangoldt step counting variable $\pi_R(x)$ and its smooth differentiable analogue $p^{-1}(x)$.

Close approximation of Diophantine equation solutions

The equation

$$Fx = f'^T(x)fx,\ fx = [f_1(x), f_2(x)]^T,\ f: D_f \subset \mathbb{R}^n \to \mathbb{R}^m,\ x \in \mathbb{R}^n,\ n > m,$$

The following is a citation of an autoregularized Gauss-Newton technique [6,7]: on par with other potential approaches to the problem

The equation

$$x^0 \in D_f,\ \varepsilon_0 > 0,\ (f'(x^k)f'(x^k) + \varepsilon_k I)(x^{k+1} - x^k) = -Fx^k,\ k = 1,2,\dots$$

$$\varepsilon_k = \frac{1}{2}\left(\sqrt{\tau_k^2 + 4c\rho_k} - \tau_k\right),$$

$$\tau_k = \|f'^T(x^k)f'(x^k)\|_\infty\ \rho_k = \|Fx^k\|_\infty,\ c = \frac{\varepsilon_0 + \varepsilon_0\tau_0}{\rho_0},$$

The SVD method, as described by Werner (2015), and the uniform vector or matrix norm, where ‖·‖_∞, are also mentioned. proceeds with the linear problem by making an assumption. Each and every domain-specific solution D_f may be found by multiplying the vector Fx^k with the local root extractor

$$'e_j\left(x,\vec{x}^{(j)}\right)=\frac{1}{1-\exp\left(-\|x-\vec{x}^{(j)}\|_2^2\right)}'$$

where $\tilde{x}^{-j)}$ is the j-th. For each iteration of the modified problem

$$F_Jx:=\left(\prod_{j=1}^{J}e_j\left(x,\vec{x}^{-(j)}\right)\right)Fx=0,\ J\geq 1,$$

the procedure is carried out with a fresh $Fx:=Fx$. Various values of x^0 and ε_0 are used to start the procedure multiple times for each answer. At every iteration of J increasing, the analytical computation of the derivatives $f'(x_k)$ and adaptive scaling of the matrices $f^T(x_k)f'(x_k)$ are performed. To ensure that the residual from the Diophantine equation disappears last essential step of the approach is to do a direct substitution check. Here, we'll go into further depth about two specific cases that highlight the importance of the last step in the previous procedure. The first one uses the sum of nine prime cubes to represent the prime number 5081. Two distinct answers are discovered:

$$5081=\begin{cases}2\times 2^3+3\times 3^3+5^3+2\times 11^3+13^3,\\ 3^3+3\times 5^3+2\times 7^3+3\times 11^3.\end{cases}$$

The problem has been solved using a real-Diophantine system and a computer with sixteen significant figures. The first line's unknowns 2, 3, 5, 7, 11, and 13 have been identified with 9 significant figures at residual $\|f_1(x)\|_\infty=10^{-14}$. After 36 unsuccessful efforts, a convergent process of this kind was built utilising 5634 iterations with 4 different beginning estimates x_0 and 9 different initial regularizers ε_0. The process culminates in an ideal match. It would be interesting to investigate if there is an infinite number of primes that can be represented as the sum of nine prime cubes. The second case involves finding integer x_5 and solving the equation $(x_1/x_2,x_3/x_4)$ where $x_i, i=1,2,3,4$ are prime integers. An number x_5 and a rational pair $(x_1/x_2,x_3/x_4)$ are the solutions to this specific case of the unsolved Fermat-Bache issue. The last stage of the approach is crucially demonstrated by the nonzero residual, which is

$$\left(\frac{787}{349}\right)^2-\left(\frac{457}{1049}\right)^3-5=0.002405488240$$

'which is the approximate solution obtained under conditions similar to the previous example,

$$x_1=787.000011,\quad x_2=348.99999357,\ x_3=457.00002128$$
$$x_4=1049.0000001,\quad x_5=5.0024058062$$

The aforementioned method for solving such problems is successful because it makes use of auto regularization, the SVD methodology, adaptive scaling, and well-isolated solutions to Diophantine equations. An explanation is still lacking for the degenerate situation, although a semi-local convergence theory exists for the nondegenerate case. When applied to this problem, the methods presented in [2] provide negligible outcomes.

Remark

According to [3-4], the negative solution of the 10-th Hilbert's problem is compatible with the Diophantine equation solution technique as our responses are approximations in a confined domain.

Von Koch's estimation and a differential equation

Now, suppose about this function

Assuming that x is greater than 1, the equation

$$K(x)=\frac{p^{-1}(x)-\mathrm{Li}(x)}{\sqrt{x}\ln(x)}$$

becomes 'Li(x)' The Riemann hypothesis, as stated by the von Koch estimate, is the same as saying that $K(x)$ is asymptotically constant, that is,

$$\lim_{x\to\infty}K(x)=\ \text{const.}\ \Leftrightarrow \text{RH}.$$

We may express the following differential equation for K as a function of x, given that $p^{-1}(x)$ is continuously differentiable:

$$K'(x)=-\left(\frac{1}{2x}+\frac{1}{x\ln(x)}\right)K(x)+\frac{\frac{dp^{-1}(x)}{dx}-\frac{1}{\ln(x)}}{\sqrt{x}\ln(x)}.$$

Therefore, we may replace $dp^{-1}/dx=1$ for $x\to p_n$ solve the equation

$$K'(x) = -\left(\frac{1}{2x} + \frac{1}{x\ln(x)}\right)K(x) + \frac{1}{\sqrt{x}\ln(x)}$$

The equation's general solution, denoted as

$$K(x) = \frac{\sqrt{x}}{\ln(x)} + \frac{c_n'}{\sqrt{x}\ln(x)},$$

this remains true in the case when c_n' is a constant and the terms on the right-hand side reflect different solutions to the inhomogeneous equation, the general solution to the homogeneous equation, and partial solutions to the inhomogeneous equation.

Assuming that $\hat{p}_n \sim p_n^{\alpha}$ with $\alpha > 0$, we may ignore the term $dp^{-1}/dx = (2\hat{p}_n + 1)^{-1}$ at the right-hand border $x \to (p_n + p_{n+1})/2$, as $x \to \infty$, This leaves us with simply $1/\ln(x)$ Here, the equation.

For the right-hand border $x \to (p_n + p_{n+1})/2$, as $x \to \infty$, we may disregard the term $dp^{-1}/dx = (2\hat{p}_n + 1)^{-1}$ if we assume that $\hat{p}_n \sim p_n^{\alpha}$ with $\alpha > 0$. The last remaining expression is $1/\ln(x)$. In this case, the formula

$$K'(x) = -\left(\frac{1}{2x} + \frac{1}{x\ln(x)}\right)K(x) + \frac{-1}{\sqrt{x}\ln^2(x)}$$

has to be solved.

Here we have the general solution

$$K(x) = -\frac{\mathrm{Li}(x)}{\sqrt{x}\ln(x)} + \frac{c''_n}{\sqrt{x}\ln(x)}.$$

As before, it is the sum of two solutions: one to the inhomogeneous equation (the first term below) and one to the homogeneous equation (the second term).

As x approaches infinity, we may define it as $\mathrm{Li}(x) \sim x/\ln(x)$ by substituting x for the leading component of the logarithmic integral.

$$K(x) \sim \begin{cases} \dfrac{\sqrt{x}}{\ln(x)} + \dfrac{c_n'}{\sqrt{x}\ln(x)}, & x \to p_n \\ -\dfrac{\sqrt{x}}{\ln^2(x)} + \dfrac{c_n''}{\sqrt{x}\ln(x)}, & x \to \dfrac{p_n + p_{n+1}}{2} \end{cases}$$

The general solution of the homogeneous equation may be represented as a sub leading function in the limit x→∞, while the initial terms can be seen as oscillations of $K(x)$ near the border of the intervals being analysed." The nature of the last dependence is currently unknown, although it might affect the constants c_n' and c''_n, since the primes' gap p ˆ_n is dependent on $\hat{p}_n$.

Conclusion

The prime counting function and the prime series may be invertibly interpolated by two diffeomorphisms that we have constructed, $p(x)$ and $p^{-1}(x)$, respectively. The new method really shines when used to numerical and analytical issues. If conventional approaches are unsuccessful in finding solutions to Diophantine equations, the function $p^{-1}(x)$ may be used. Furthermore, when more data on prime gaps is available, its behaviour similar to π(x) as x→∞ might potentially reveal important details regarding the asymptotic and non-asymptotic distribution of primes, perhaps improving our understanding of the Riemann hypothesis

Reference

1. Milgram, M. (2020). Research Article An Integral Equation for Riemann's Zeta Function and Its Approximate Solution.
2. Min, H., Tarmoun, S., Vidal, R., & Mallada, E. (2021, July). On the explicit role of initialization on the convergence and implicit bias of overparametrized linear networks. In *International Conference on Machine Learning* (pp. 7760-7768). PMLR.
3. Xu, L., Chatterton, S., & Pennacchi, P. (2021). Rolling element bearing diagnosis based on singular value decomposition and composite squared envelope spectrum. Mechanical Systems and Signal Processing, 148, 107174.
4. Werner, F. (2015). On convergence rates for iteratively regularized Newton-type methods under a Lipschitz-type nonlinearity condition. *Journal of Inverse and Ill-Posed Problems*, *23*(1), 75-84.
5. Mavrodiev, S. C., & Deliyergiyev, M. A. (2017). Computation of the Binding Energies in the Inverse Problem Framework. In *EXOTIC NUCLEI: EXON-2016 Proceedings of International Symposium* (pp. 330-337).